High School Competency Exams in Hong Kong, China and Teaching Training Programme

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Introduction

The present mathematics curriculum in Hong Kong, China is as follow:
Students will need to do 6 years of primary education and 5 years of secondary education. After the secondary education, students will do a 2 years Advanced –Level education. The following table describe the present curriculum structure in Hong Kong, China. However, a new secondary curriculum take effect from 2009, and by the year 2012, all students in Hong Kong, China will do a 6 year secondary programme (3 year junior secondary and 3 year senior secondary).

Duration	Subjects	
2 years	(A-Level , 2 years) Pure Mathematics Applied mathematics	AS Level, 1 years Applied Mathematics Mathematics and Statistics
5 years	Secondary Mathematics (taken by all students) Additional Mathematics (usually taken by students in science stream)	
6 years	Primary mathematics	

A pass grade in Secondary Mathematics is a requirement for students going to A-Level studies. Hence all Hong Kong, China students need to take this subject. And the discussion of the achievement in this subject may serve the requirement in discussing the standard of High School Graduation/Competency Exams in Hong Kong, China.

For Secondary Mathematics, around 60,000 to 68,000 students each year (day school first attempt) take this paper. From 1999 to 2009, the passing rate of Secondary Mathematics is around 71% to 76%. However, there is no information on the pass mark of getting a pass grade.

For Additional Mathematics, the number of students (day school first attempt) taking this papers is around 17500 to 18500. From 1999 to 2009, the passing rate of Additional Mathematics is around 81% to 85%.

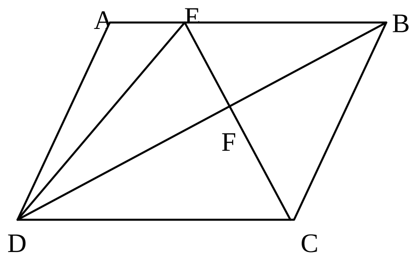
The Secondary Mathematics paper consisted two parts, part one is the multiple choices and part 2 is conventional questions. In part 1, there are 54 multiple choice questions. It is further divided into two sections. Section A consist of 36 questions, which are more fundamental and section B consist of 18 questions which is more difficult. Usually, more than 70% of the students can correctly answer 18 of the 54 questions. Other more difficult questions serve to discriminate the standards of the students.

The following are some examples of attempt by students in Secondary Mathematics.

For example, in one year, 60 % of the students could not see or use the property “two triangles has the same base and same height have the same area”

Question for concept:
In the figure, ABCD is a parallelogram;
E is a point on Line AB.
If EC and BD intersect at F,
find the ratio of the area

of ⊿DEF and ⊿CBF

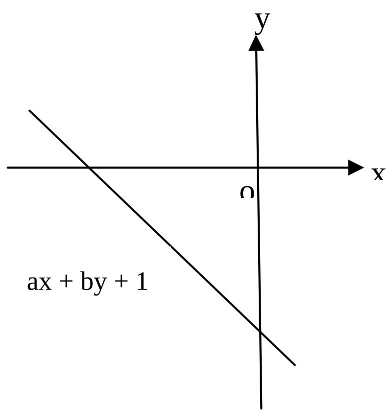


Of the 4 options, only 40% of the students chose the correct option “1:1”. Many students did not see that⊿CDE and⊿BCD has the same base and same height.

Question for concept and procedure:
The figure describe the straight line

ax + by + 1 = 0。

.which of the following are correct？



The correct option (A) “a > 0 and b > 0” has a response percentage of 35%. However, the incorrect option (D) $a < 0$ and $b < 0$ has a very high response percentage of 33%.

Example 3

Students are not familiar with transformation of structure. For example, when Reminder Theorem is examined in a way that involves transformation, only 19% of them could answer correctly.

Question for mathematical thinking: If f(x) is a polynomial, and f(x) is divisible by (x-1), then which of the following is a factor of f(2x+1).

The correct option with answer "x" has only a response of 19%. The highest response percentage of a wrong option "2x-1" is 35%. Other incorrect option like "2x+1" attract 27% of the reply.

The program for promoting good Mathematics teaching in Hong Kong, China

In order to promote good practice in mathematics teaching, the Education Bureau of Hong Kong has been organizing a lot of in-service training programme for teachers. Some of them are offered by University and some are by the Bureau staff.

One of the project is lesson study, a project holds in Hong Kong Institute of Education. Primary school teachers can work with staff in University to establish a framework for lesson observation and improve their mathematics teaching. The programme has lasted for 5 years and responses from school are positive.

Another project at the Hong Kong Institute of Education is the study of assessment system, helping school to monitoring their system of assessment in subject including mathematics. This helps school teachers to establish assessment items in mathematics.

A third system project in Hong Kong, China is the invitation of teachers from Mainland China to come to Hong Kong, China and teach for half year. Their planning of teaching is discussed and their teaching can be observed. These teacher also serve as "resources teachers" to help the local school teachers to establish an effective teaching plan.

A fourth system programme organized by the Education Bureau is to assigned staff to support local school to develop school based activities. Each year, about 20 schools in Hong Kong, China are involved in the project. The support staff will work with school teachers to form some school based teaching activities and also hold discussion to ensure teaching a topic at different level effectively.

A fifth system programme is a 5-weeks teacher training programme, teachers come to the Institute to study technique of mathematics teaching and developing materials doing in-depth discussion on teaching selected topics, and also a brief knowledge of most recent research results and information related to high school mathematics teaching. A teacher should be able to develop handful of related materials.

The 5 weeks programme

For a long time, teachers are trained to teach according to what is in the mathematics textbook, and students are expected to answer question within the textbook content. For those mathematics questions that did not appear in the textbook, they are usually ignored.

The 5-weeks programme enjoy a lot of positive responses from teachers, as the course not only provide theoretical framework, but also practical technique of teaching and development of teaching material. The in-service teachers attended the programme and learn the latest technique or skills to teach certain topics in mathematics. If a program can satisfy the needs of teacher to allow them to develop their own material, it enjoy more positive support, especially that the topics that teachers think difficult to teach are taken for discussion and teachers benefit from it. They may use new approaches to teach a certain topic, or they learn to use various different approaches to teach the same topic.

However, the technique wise training is not always effective in teacher education. It is more important to change the mindset of the teachers so that not only they accept the technique and they are ready to use such technique, but also that they can reproduce some of the technique themselves. If they could not produce teaching technique of their own, then we will need teacher retraining again and again, making such retraining programme not very effective.

The present situation of mathematics teaching - generalist and specialists

A mathematics teacher should be able to synthesis knowledge of mathematics, developmental and psychological theories, and the pedagogical knowledge for teaching the subjects. So, a teacher needs training in both mathematics knowledge and curriculum theories in teaching mathematics.

In Hong Kong, China, most of the mathematics teachers are math trained. But for teachers at the lower forms, many are graduates with an engineering degree or even not trained in mathematics. What they need is the specific technique in teaching the topic, and most of them are included in the textbook.

Knowing the technique of teaching mathematics is not enough for good mathematics education. Teachers should be able to do exploration in mathematics if they wish to enhance their teaching. By doing exploration, teachers have the ability to know the structure, allow teachers to construct problems of mathematics.

There are in general two kinds of mathematics teachers, the specialist and the generalist. They can be divided into three type of mathematics teaching, transmission type, discovery type and connectionist type. Transmission type is those who worked with a standard procedure in calculation, discovery type is those who wish to emphasis on procedure that are practical and can be discovered by students. Connectionist is those who try to connect what is learned by students and the content of mathematics. By exploring the content of some mathematics investigation, teachers understand the process of their own thinking. Connectionist types are more likely to be highly effective teachers.

	Transmission approach	Discovery approach	Connection approach
Generalist training	Very Likely	Less Likely	Very unlikely
Specialist training	Very Likely	Likely	Likely

Those teachers who are generalist are usually transmission type than discovery type. This is because they may not be able to understand the structure of mathematics and the structure of the problem and solution. And direct teaching is a safe channel.

The other extreme end of the generalist approach is what we now call “facilitating learning”, in which teacher is a facilitator and did not teach. The results of such teaching may be disastrous, as the term “facilitator” helps those generalists to hide from doing real knowledge in mathematics or not even prepare their teaching.

For that teacher with specialist training in mathematics, they can choose to use transmission approach. However, they are more likely to use discovery approach as they could use their mathematics knowledge to guide students to do discovery work.

Examples that specialist training could use connection approach

The first step is for teachers to get good and interesting mathematics questions. And from then they can obtain some basic technique in solving the mathematics problem. And the last step is for them to observe and obtain the structure of the questions, and relate the structure to the solution. For example, the teaching of the following mathematics structure.

Question：

Express the fraction $\frac{n-1}{n}$ as a sum of 3 fractions.

$\frac{n-1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

Why the question? Teacher can easy find connected mathematics problem.

Question：

An old man has 17 horses and he is going to leave $\frac{1}{2}$ of his horses to his elder son, $\frac{1}{3}$ of his horses to his second son, $\frac{1}{9}$ of his horses to his youngest son.

The answer is possible when one horse is borrowed to make up to 18. And it is because of the following mathematical structure, $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{17}{18}$, which has the same structure as $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{n-1}{n}$.

To show the general solution, we know that as $\frac{n-1}{n} < 1$, a ≥ 2. And a can only be 2.

As the sum of the 3 fractions is less than 1, one of them should be greater than $\frac{1}{3}$, and hence $\frac{1}{2}$ is one of the three fractions.

Take $\frac{1}{2}$ as one of the 3 fraction, then we have $\frac{n-1}{n} = \frac{1}{2} + \frac{1}{b} + \frac{1}{c}$.

To maximize the sum of the 3 fractions, we have $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{47}{60}$.

Using trail and error, $\frac{n-1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{c}$ and obtain $\frac{1}{c} = \frac{1}{6} - \frac{1}{n}$。

As n is greater than c, so $\frac{1}{c} = \frac{1}{6} - \frac{1}{c}$. That is, c ≤ 12. As $\frac{1}{c}$ is less than $\frac{1}{6}$. So we have c ≥ 7.

We have the following 7 answer for $\frac{n-1}{n} = \frac{1}{2} + \frac{1}{b} + \frac{1}{c}$,

$\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, $\frac{11}{12} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6}$, $\frac{11}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{12}$.

$\frac{17}{18} = \frac{1}{2} + \frac{1}{3} + \frac{1}{9}$, $\frac{19}{20} = \frac{1}{2} + \frac{1}{4} + \frac{1}{5}$, $\frac{23}{24} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8}$.

$\frac{41}{42} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7}$.

Further Exploration 1：

Write $\frac{n+1}{n}$ as sum of three unit- fractions

$\frac{n+1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

Further Exploration 2：

Write $\frac{n-1}{n}$ as sum of four unit-fractions

$\frac{n-1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$。

Example in applying technique of solving problem

For teachers to be able to connect his lessons, he is able to found example which extend the mathematics structure. For example, the technique for solving the following question is basic in high school, but extending the application needs more different type of questions

First, observe the process of solution of the following problem

$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\frac{1}{56}+\frac{1}{72}$

$= \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\frac{1}{5\times 6}+\frac{1}{6\times 7}+\frac{1}{7\times 8}+\frac{1}{8\times 9}$

$= (1-\frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) + (\frac{1}{4}-\frac{1}{5}) + \ldots + (\frac{1}{8}-\frac{1}{9})$

$= \frac{8}{9}$.

Exploration 1：

By using the above process, find a structure and solve the following question:

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\frac{1}{7\times 9}+\frac{1}{9\times 11}+\frac{1}{11\times 13}+\frac{1}{13\times 15}+\frac{1}{15\times 17}$$

The idea of splitting the terms and then cancelled can be extended to other examples.

Exploration 2：

Simplify $\frac{1}{\sin 2\alpha}$ + $\frac{1}{\sin 4\alpha}$ + ….+ $\frac{1}{\sin 2^n\alpha}$。

As $2\cos^2\theta - \cos(2\theta) = 1$, hence $\frac{1}{\sin 2\alpha} = \frac{2\cos^2\alpha - \cos 2\alpha}{\sin 2\alpha} = \frac{2\cos^2\alpha}{\sin 2\alpha} - \frac{\cos 2\alpha}{\sin 2\alpha} = \frac{2\cos^2\alpha}{2\sin\alpha\cos\alpha} - \frac{\cos 2\alpha}{\sin 2\alpha}$

$= \cot\alpha - \cot 2\alpha$。

Similarly, $\frac{1}{\sin 4\alpha} = \cot 2\alpha - \cot 4\alpha$,

$\frac{1}{\sin 2^n\alpha} = \cot 2^{n-1}\alpha - \cot^{2n}\alpha$。

Hence $\frac{1}{\sin 2\alpha} + \frac{1}{\sin 4\alpha} + \ldots + \frac{1}{\sin 2^n\alpha}$

$= \cot\alpha - \cot 2\alpha + \cot 2\alpha - \cot 4\alpha + \ldots + \cot 2^{n-1}\alpha - \cot^{2n}\alpha$

$= \cot\alpha - \cot^{2n}\alpha$。

Reverse process, combining terms

Exploration 3：

Find the value of sin10° sin30° sin50° sin70°.

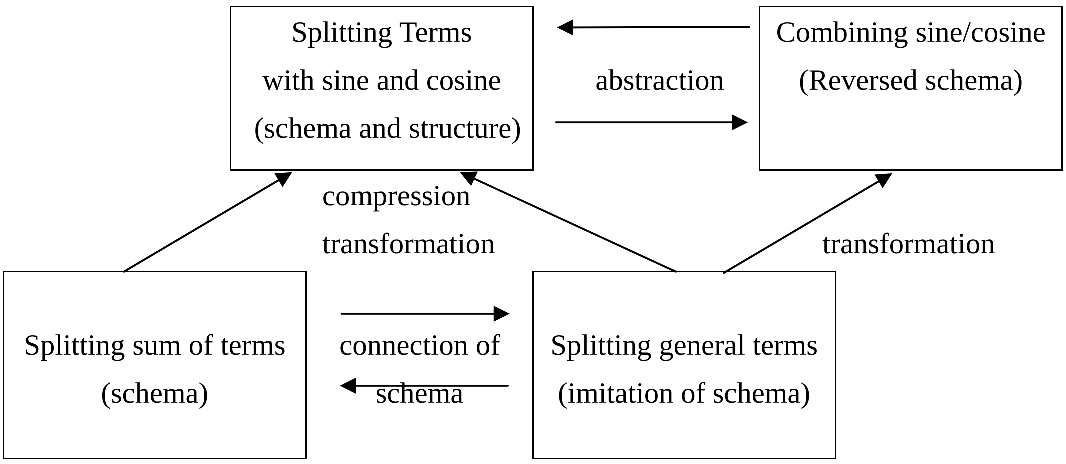
By the relation of sine and cosine, try using cos10° cos30° cos50° cos70°
And make use of the formula sinAcosA.
Let A = (sin10° sin30° sin50° sin70°), B = (cos10° cos30° cos50° cos70°)

then $AB = \frac{1}{16}(\sin 20° \sin 60° \sin 100° \sin 140°)$

$= \frac{1}{16}(\cos 10° \cos 30° \cos 50° \cos 70°) = \frac{1}{16}B$。

As $B \neq 0$, $A = \frac{1}{16}$。

The process of exploration in the examples of using technique of splitting terms can be described by the following framework.



An example used in the 5-week programme in mathematics teacher training
The following is some examples in the programme, Finding the area of triangle and quadrilateral.

Teachers will analyze with their class the property of the area of a triangle.

If $S_{(a,b,c)}$ denote the area of a triangle with sides a, b, and c, then there are properties that need to go through with students.

We will have $S_{(a,b,c)} = S_{(b,c,a)} = S_{(c,a,b)}$. This is the symmetric property.

Also, as the triangle is not biased with any three sides, the formula of the area of the triangle should include symmetrical expressions such as $a+b+c$ (degree one) , $a^2+b^2+c^2$,

$ab+bc+ca$ $(a+b+c)^2$ (degree two), (a+b+c)(ab+bc+ca) (degree three) etc.

Also, the dimension of the area of a triangle must be the square of length.

Discussion with the class come to the summary that, the formula of area of a triangle could look like this:

$S_{(a,b,c)} = K\sqrt{(a+b+c)(a^3+b^3+c^3)}$, as it is symmetrical and satisfy the dimensional requirement.

However, another important property of a triangle is the "sum of any two sides is longer than the third side", so there should be a factor in the form of b + c – a > 0. And b + c – a = 0 is a boundary condition. If a + b = c or b + c = a or c + a = b, then the area of the triangle will be zero. Hence $S_{(a,b,c)}$ consisted the three factors (a + b –c), (b + c – a), and (c + a – b).

Perhaps the formula is $S_{(a,b,c)} = K[(a+b-c)(b+c-a)(c+a-b)]^{\frac{2}{3}}$, but the formula consists of only three factors and it will give negative values of a area, such as a = 1, b = 2, and c = 100. and such triangle does not exist.

Hence the guess is that another factor $(a+b+c)$ exist, and the formula is in the form $S_{(a,b,c)} = K(a+b+c)(b+c-a)(c+a-b)$. According to the requirement of dimension, it is rewritten as $S_{(a,b,c)} = K\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}$ 。

Here, the conjectured formula satisfies our requirement in symmetry, dimensional and boundary condition. And we test to find out the numerical values of K. By putting in the information of a right angle triangle, we have $c^2 = a^2 + b^2$, and the area $\frac{1}{2}ab$.

Then

$$K\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}$$

$$= K\sqrt{[(a+b+c)(a+b-c)][c+(b-a)][c-(b-a)]}$$

$$= K\sqrt{\left[(a+b)^2-c^2\right]\left[c^2-(b-a)^2\right]}$$

$$= K\sqrt{\left(a^2+b^2+2ab-c^2\right)\left(c^2-b^2+2ab-a^2\right)}$$

$$= K\sqrt{2ab \times 2ab}$$

$$= 2Kab$$

By $\frac{1}{2}ab$ = 2Kab, we have K=$\frac{1}{4}$. Hence the formula is deduced to be

$$S_{(a,b,c)} = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}$$

The above deduction is based on assumption and conditions, and need to be proved.

If we checked the validity of the formula again, using the data of a equilateral triangle with side a, and area $\frac{\sqrt{3}}{4}a^2$, the area given by the formula is $\frac{1}{4}\sqrt{3a \times a \times a \times a}^2 = \frac{\sqrt{3}}{4}a$, which in line with fact.

However, a mathematical proof is more than verification with examples. The following is the proof of the formula, though the proof itself is not the same as it was derived 2000 year ago.

Using area = $\frac{1}{2}$ ab sinC to device the formula of area of a triangle with sides a, b and c.

We start with a triangle ABC and $S_{(a,b,c)} = \frac{1}{2}$ ab sinC.

Squaring, $S^2_{(a,b,c)} = \frac{1}{4}a^2b^2\sin^2C = \frac{1}{4}a^2b^2(1-\cos^2C)$。

By Cosine Rule, $\cos C = \frac{a^2+b^2-c^2}{2ab}$, after substitution,

$$S^2_{(a,b,c)} = \frac{1}{4}a^2b^2\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{4a^2b^2}$$

$$= \frac{1}{16}\left[\left(2ab + a^2 + b^2 - c^2\right)\left(2ab - a^2 - b^2 + c^2\right)\right]$$

$$= \frac{1}{16}\left[(a+b)^2 - c^2\right]\left[c^2 - (a-b)^2\right]$$

$$= \frac{1}{16}(a+b+c)(a+b-c)(c+a-b)(c+b-a)$$

If we put $p = \frac{1}{2}(a+b+c)$, then $\frac{1}{2}(a+b-c)$ = p – c , $\frac{1}{2}(b+c-a)$ = p – a , $\frac{1}{2}(c+a-b)$ = p – b。

$\Rightarrow S^2_{(a,b,c)}$ = p(p-a)(p-b)(p-c)

$\Rightarrow S_{(a,b,c)} = \sqrt{p(p-a)(p-b)(p-c)}$

The formula $S_{(a,b,c)}$ now satisfy the requirement that：

(i) Area are positive, $S_{(a,b,c)} \geqq 0$。

(ii) Formula in area of triangle is symmetric, $S_{(a,b,c)} = S_{(a,b,c)} = S_{(c,a,b)}$

(iii) Dimension of area $= L^2$

(iv) Boundary conditions, when $a+b=c$, $b+c=a$, $c+a=b$, $S_{(a,b,c)} = 0$

(v) Proportion of area, $S_{(\lambda a,\lambda b,\lambda c)} = \lambda^2 S_{(a,b,c)}$, $\lambda \geqq 0$。

The discussion continues to explore the relation of the Heron's Formula and the Pythagoras theorem.

Exploration

If right angle triangle with two adjacent sides a, b, show that we can obtain the relation $c^2 = a^2 + b^2$.

By using the formula of area and also it equals to $\frac{ab}{2}$, that is

$\frac{1}{16}(a+b+c)(a+b-c)(c+a-b)(c+b-a) = (\frac{ab}{2})^2$, students are required to deduce the result $c^2 = a^2 + b^2$.

In China, around 1000 years ago, a scholar "Zhun Kiu Siu" deduce a formula for the area of a triangle in his book "Nine Chapters in Mathematics" The formula is

$$S_{(a,b,c)} = \frac{1}{2}\sqrt{b^2a^2 - \left(\frac{b^2+a^2-c^2}{2}\right)^2}$$

Exploration

Deduce the Heron's formula from the Zhun's formula.

From $S^2_{(a,b,c)} = \frac{1}{4}\left[b^2a^2 - \left(\frac{b^2+a^2-c^2}{2}\right)^2\right]$

$= \frac{1}{16}\left[(2ab)^2 - (b^2+a^2-c^2)^2\right]$

$= \frac{1}{16}\left[(b+a)^2 - c^2\right]\left[c^2 - (b-a)^2\right]$

$= \frac{1}{16}(a+b+c)(a+c-b)(b+c-a)(b+a-c)$

$= p(p-a)(p-b)(p-c)$

There are many possible ways to deduce the formula of area of triangle by using nowadays technique.

Exploration

Deduce the formula for area of triangle from $\sin A = \dfrac{2\Delta ABC}{bc}$, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ and $\sin^2 A + \cos^2 A = 1$.

We substitute $\sin A = \dfrac{2\Delta ABC}{bc}$ and $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ into $\sin^2 A + \cos^2 A = 1$, then

$$\frac{4(\Delta ABC)^2}{b^2c^2} + \frac{(b^2 + c^2 - a^2)^2}{4b^2c^2} = 1。$$

$$(\Delta ABC)^2 = \frac{1}{16}(4b^2c^2 - (b^2 + c^2 - a^2)^2)$$

$$= \frac{-1}{16}(b^2 + c^2 - a^2 - 2bc)(b^2 + c^2 - a^2 + 2bc)$$

$$= \frac{1}{16}[(b + c)^2 - a^2][a^2 - (b - c)^2]$$

Another approach can be used as follow:

$$\Delta^2 = \frac{1}{4}b^2c^2\sin^2 A$$

$$= \frac{1}{4}b^2c^2(1 - \cos^2 A)$$

$$= \frac{1}{4}b^2c^2(1 - \cos A)(1 + \cos A)$$

By $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, we have

$$1 + \cos A = \frac{b^2 + c^2 - a^2 + 2bc}{2bc} = \frac{(a + b + c)(b + c - a)}{2bc}$$

$$1 - \cos A = \frac{b^2 + c^2 - a^2 - 2bc}{2bc} = \frac{(a - b + c)(a + b - c)}{2bc}$$

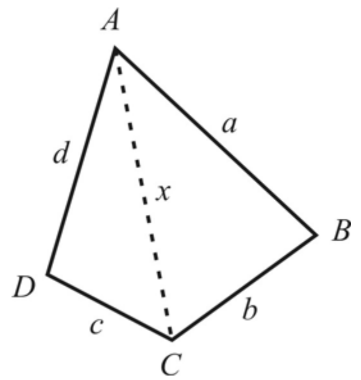
Hence $\Delta^2 = \dfrac{1}{16}(a + b + c)(b + c - a)(c + a - b)(a + b - c)$

Exploration

ABCD is a quadrilateral, AB = a, BC = b, CD = c, DA = d。

Show that the area can be

$$S_{ABCD} = \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) - \frac{1}{2}abcd\cos(B+D)$$



The process of discussion:

AC = x, using cosine rule,

$$a^2 + b^2 - 2ab\cos B = x^2 = c^2 + d^2 - 2cd\cos D$$

$$\Rightarrow\ ab\cos B - cd\cos D = \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)$$

$$\Rightarrow\ (ab\cos B - cd\cos D)^2 = \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2$$

$$\Rightarrow\ (ab\cos B)^2 + (cd\cos D)^2 = \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2 + 2abcd\cos B\cos D$$

$$S^2_{ABCD} = \left(\frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D\right)^2$$

$$= \frac{1}{4}\left((ab\sin B)^2 + (cd\sin D)^2 - 2abcd\sin B\sin D\right)$$

$$= \frac{1}{4}\left((ab)^2(1-\cos^2 B) + (cd)^2(1-\cos^2 D) + 2abcd\sin B\sin D\right)$$

$$= \frac{1}{4}\left((ab)^2 + (cd)^2 - \left(\left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2 + 2abcd\cos B\cos D\right) + 2abcd\sin B\sin D\right)$$

$$= \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) - \frac{1}{2}abcd(\cos B\cos D - \sin B\sin D\right)$$

$$= \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) - \frac{1}{2}abcd\cos(B+D)$$

B and D are two angle of the quadrilateral. The result can apply to convex and concave quadrilateral.

For the above area to achieve maximum value, cos(B+D) = -1, That is B + D = 180°, which means ABCD is a quadrilateral inscribed in a circle.

Hence for a inscribed quadrilateral,

$$S^2_{ABCD} = \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) + \frac{1}{2}abcd$$

Exploration:

We know that a triangle with sides a, b, c has area =

$\sqrt{p(p-a)(p-b)(p-c)}$, where p is half the perimeter of the triangle.

Now for an inscribed quadrilateral ABCD, AB = a, BC = b, CD = c, DA = d,

would it be possible that the area = $\sqrt{(p-a)(p-b)(p-c)(p-d)}$, where p is half the perimeter of the quadrilateral.

That is , will

$$S_{ABCD} = \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) + \frac{1}{2}abcd$$

$= \sqrt{(p-a)(p-b)(p-c)(p-d)}$?

Mathematical thinking will first check whether the formula of the area of the quadrilateral $\sqrt{(p-a)(p-b)(p-c)(p-d)}$ can hold at a special case d = 0, which is a triangle.

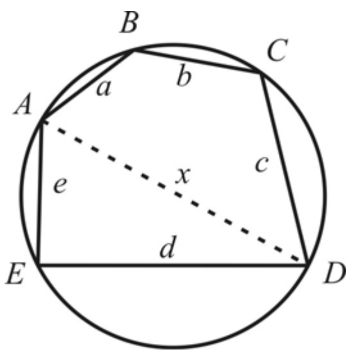
As $S_{(a,b,c,0)} = \sqrt{(p-a)(p-b)(p-c)(p-0)} = \sqrt{p(p-a)(p-b)(p-c)}$ 。

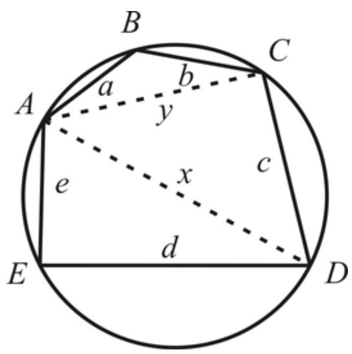
Hence the formula holds for the special case. The next step is to deduce that the formula is correct. This will be an exploration for the students.

The exploration can carry on to the case of area of a pentagon.

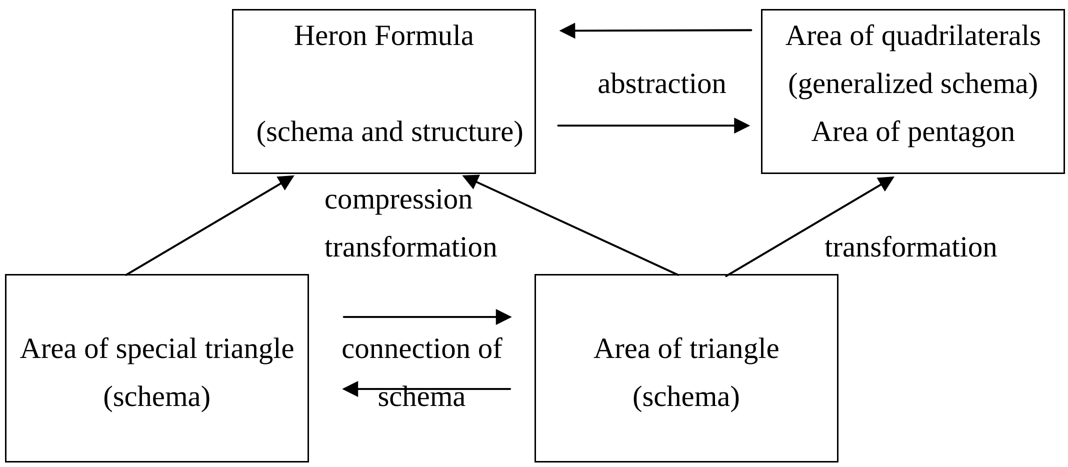
Exploration

Can we find the area of a pentagon through dissecting the figure into a quadrilateral and a triangle.





The process of exploration in the examples of area of triangle and quadrilateral



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