REASONING ABOUT THREE-EIGHTHS: FROM PARTITIONED FRACTIONS TOWARDS QUANTITY FRACTIONS

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Curriculum documents describe the importance of questioning, reasoning and reflecting as contributing to Working Mathematically. A research lesson on the development of units of different sizes (eighths) associated with measurement and fractions, is developed as a vehicle for developing mathematical reasoning through argumentation in a composite Year 3–4 class. Making a transition from embodied fractions (parts of a whole) to recognising the equal whole needed for comparison of fractions as mathematical objects extends the current Mathematics curriculum in New South Wales. The lesson study also highlighted the need to develop taken as shared meaningfor fraction units used in classroom argumentation.

REPRESENTATIONAL CONTEXTS

Working Mathematically draws on ways on ways of seeing, questioning, interpreting, reasoning and communicating. This type of mathematical thinking was summarised by Schoenfeld as follows:

> Learning to think mathematically means (a) developing a mathematical point of view— valuing the process of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of understanding structure—mathematical sense-making. (Schoenfeld, 1989, as cited in Ball, 1993, p. 157)

The tools used in the service of understanding structure are often derived from *models of* the mathematics. For example, partitioned circles or rectangles are used as regional models of partitioned fractions (Watanabe, 2002) which can contribute to associated concept images of fractions . Yet physical models have limitations in that they at best imply the mathematical concept, and can add unwarranted components to the intended concept image. In particular, students' evoked fraction concepts suggest that equality of area is not always the feature abstracted from regional models used in teaching fractions (Gould, 2005). Further, to be able to interpret the part -whole comparison of area intended by the regional model, children need to be familiar with the context, which in this example includes the concept of area (Lampert, 1989). Choosing an appropriate representational domain is an important teaching decision in developing students' mathematical understanding.

In practice, fractions exist in essentially two forms: embodied representations of comparisons, sometimes called partitioned fractions, and mathematical objects, also known as quantity fractions. A *partitioned fraction* (Yoshida, 2004) can be described as the fraction formed when partitioning objects into *b* equal parts and selecting *a* out of *b* parts to arrive at the partitioned fraction *a/b*. A partitioned fraction can be of either discrete or continuous objects but a partitioned fraction is always a fraction of something. By comparison, quantity fractions are mathematical objects defined as fractions that refer to a universal unit. Asking the question, which is larger, one-half or three-eighths, only makes sense if the question is one of quantity fractions. Quantity fractions implicitly reference a universal unit, a unique unit-whole, which is independent of any situation. If one-half and three-eighths as mathematical objects do not refer to a universal whole, we cannot compare them.

The learning of fractions is subject to a paradox that is central to mathematical thinking (Lehrer & Lesh, 2003). On one hand, a fraction such as 2/3 takes its meaning from the situations to which it refers (partitioned fraction); on the other hand, it derives its mathematical power by divorcing itself from those situations (quantity fraction). Working with partitioned embodiments of the fraction "3/7" can elicit a parts-of-a-whole meaning as "three out of seven", but without divorcing the fraction notation from this context interpreting "7/3" does not make sense. It is difficult for students to become aware of a unit-whole when the unit-whole is often implicit in everyday situations involving fractions. To make the transition from partitioned fractions to quantity fractions, students need to develop a sense of the size of fractions. A sense of the size of fractions is what Saenz -Ludlow (1994) refers to as conceptualising fractions as quantities. Mathematical thinking associated with working with units of various types and in particular, the introduction of abstract units, are central to mathematics.

PLANNING THE LESSON

In the dominant instructional model used to teach fractions in New South Wales, children learn to divide objects into equal parts. Next, children learn to count the number of parts of interest and place the result of this count above the count of the total number of parts using the standard fraction notation. This part-whole recording method is used to introduce fraction notation. The transition from counting parts of a model to recording fraction notation is followed by instruction on the traditional algorithmic manipulation of the whole number components of the fraction notation, known as operating with fractions.

Unfortunately, the predominant instructional model is not successful for many students and fractions are a particularly troublesome area of the elementary mathematics curriculum in NSW. The language associated with fractions in English contributes to a number of misconceptions for students. Unlike most Asian languages, English uses the same terms for naming ordinals and fractions (e.g. third, sixth, ninth). It is also easy for students to not hear the soft sounds at the end of fraction names, which can lead to confusion between whole numbers (e.g. six) and fractions (e.g. sixth). Thus, although six sixes are thirty -six, six sixths are one.

Following the mathematics syllabus, children describe halves in everyday contexts in their first year of school, Kindergarten. Ironically, everyday representational contexts for halves include examples such as cutting a piece of fruit into halves, where the means of determining the equality of the pieces relies on an understanding of volume. In the following two years (Years 1 and 2) children are expected to model and describe a half or a quarter of a whole object or collection of objects as well as to use the fraction notation 1/2 and 1/4. In Years 3 and 4 children model, compare and represent fractions with denominators 2, 4 and 8 as well as find equivalence between halves, quarters and eighths.

For fractions to make the transition from embodied partitions to mathematical objects the idea of a universal unit-whole needs to be established. This universal unit-whole is a ‘one’ that remains the same size in all contexts and is similar to a standard unit of measure. Making a transition from embodied fractions (parts of a whole) to recognising the equal whole needed for comparison of fractions as mathematical objects is the unit goal for lesson study outlined below. The idea of lesson study was new to the teacher and the school. Further, the idea of the need to identify the equal whole and the specific role of representational contexts are not part of current teaching practice in elementary schools in New South Wales. Consequently, the planned abstraction referred to in the unit goal is a very ambitious goal for the composite Year 3 and Year 4 class taking part in the lesson study.

Developing thinking through argumentation

The key window for considering mathematical thinking in this lesson study is through justification in reasoned argument. Learning to argue about mathematical ideas is fundamental to understanding mathematics. Palincsar and Brown (1984) wrote that “ ...understanding is more likely to occur when a child is required to explain, elaborate, or defend his position to others; the burden of explanation is often the push needed to make him or her evaluate, integrate and elaborate knowledge in new ways.” Argument here is taken to mean a discursive exchange among participants for the purpose of convincing others through the use of mathematical modes of thought.

The ways in which students seek to justify claims, convince their classmates and teacher, and participate in the collective development of publicly accepted mathematical knowledge contribute to mathematical argument. In a culture that expects student understanding, teaching mathematics is more than merely telling or showing students; teachers must enable students to create meanings through their own thinking and reasoning. Classroom argumentation needs opportunities to move from authority-based arguments (because the teacher says so or the text states this) to reasoning with mathematical backing. That is, “how do you know?” is the key question. The expectation is that students arrive at consensus through reasoned argument, reconciling different approaches through demonstration using a common model.

THE LESSON: THREE-EIGHTHS OF THE BOARD

The link between the process of division and the creation of fractions is not always clear to students. To simplify the creation of this link, the attribute of length is used instead of area to create partitioned fractions. Although regional models are often used to introduce fractions, some students focus on the number of regions and not the area of the regions compared to the whole shape in abstracting the fraction relationship.

By the start of Year 3, children can model and describe a half or a quarter of a whole object or collection of objects as well as being familiar with the fraction notation 1/2 and 1/4 (syllabus reference NS 1.4). Multigrade classes are quite common in New South Wales and the class taking part in the lesson study described here had 6 Year 3 students and 21 Year 4 students. The Mathematics K–6 syllabus describes content in stages corresponding to two school years with the exception of the Kindergarten year, referred to as Early Stage 1. As the students in the study had covered the fractions content from Stage 1 (Years 1 and 2) this lesson was designed as an introduction to eighths and the relationship between eighths, quarters and half (Appendix A).

The lesson started by inviting three students to estimate where three-eighths of the width of the class white-board would be, mark the point and put their initials next to their estimates. The students were chosen by the teacher based on her knowledge of the students to obtain variation in the estimates. The fraction 3/8 was used with the attribute of length to focus on composition of partitioning through repeated halving. As the students had previously covered work on 1/2 and ¼ the use of 3/8 also provided scope for iterating the unit fraction. Having the students record their initials next to the estimates gains ownership of the estimate by the student and encourages a desire to find out. It also makes discussion about the estimates easier by providing a name for the estimate.

A piece of string the same length as the white-board was cut and used by students to form one-eighth and in justifying which estimate was closest. The class discussion also provided opportunities to relate the values of half, quarters and eighths of the same length. The students returned to their desks following the discussion and located positions corresponding to various numbers of eighths of different sized intervals (see Appendix B). After discussing the location of three-eighths on different sized intervals the final question was posed: *Could 4/8 ever be less than 2/8?*

Students' explanations

Having established the unit of one -eighth, the teacher used one -half of the length of the whiteboard as a benchmark to determine how close the estimates were to three-eighths. The teacher asks how close was Jack's estimate to three-eighths and a very rapid exchange takes place between the students. One student (Charlotte) says that Jack was not as close as Emily, a mark greater than one-half of the length of the board. Another student thinks about the response and quickly says, "That's a half". The teacher picks up on the exchange.

Teacher: *That's an interesting comment. Charlotte's comment was he wasn't as close as Emily.*

Teacher: *Remember we were wanting to find three-eighths of the length of the white-board. If this is halfway, how much of the white-board do you think Emily found?*

Stephen: *Two, three quarters, two and a half quarters.*

Teacher: *Two and a half quarters? OK. Amy, how...*

Amy: *Four and a half eighths.*

Teacher: *Jessica.*

Jessica: *It could be five-eighths.*

Jessica then went on to describe the size of the unit (partitioned) fraction one-eighth and that one half of the board corresponds to four-eighths (Fig. 1), so adding on one eighth more making five-eighths would be a position very close to Emily's mark.

Figure 1. Describing one-half as four-eighths

The discussion arising from using the benchmark of one-half to determine which of the estimates was closest to three-eighths, and the subsequent description of each estimate location in terms of eighths, was very informative. The orchestration of the discussion did enable the teacher to intervene to seek justifications for the different beliefs as to which estimate was closest to three-eighths.

However, developing *taken as shared* meaning for fractions as quantities is not simple. In a later part of the lesson, when one student explains why three-eighths is at a different location on the first two intervals, he suggests that it might be because "all of us used different strategies to work out the answer". This suggested was followed by a surprising exchange.

Teacher: *What strategies did you use?*

Student: *I cutted them up into all kinds of different quarters and then I went to... uhm, and then I counted it from one for every quarter.*

Teacher: *Why did you cut it up into quart ers?*

Student: *Because ... uhm...* [Pause]É *I cut it up into different quarters because then I'd know how much of each part of it is the same.*

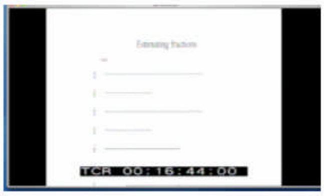
The exchange is puzzling for both the student and the teacher, as they appear to have different meanings for quarters. When students are challenged to explain their reasoning the evoked concept images (Tall & Vinner, 1981) can reveal unexpected interpretations of fractions such as quarters. The evoked image related to the term quarter was quite different between student and teacher. The student was using the term quarter for a general fraction part. Rather than meaning one of four equal parts comprising the whole, the term quarter meant 'equal piece' for this student. This interpretation was confirmed when the video of the lesson was replayed to the student. Just as the common use of fraction terms can lead to exchanges that suggest that one-half can be bigger than another half (*You take the bigger half)* the same appears to occur with the term quarter. As the teacher and the student did not have a *taken as shared meaning* for a 'quarter' the discussion did not advance the understanding of fraction units. Recognising that the nature of mathematical argument may vary between cultures (Sekiguchi & Miyazaki, 2000) confronting the individual's misconception about fraction units was not always straightforward. The teacher can only respect a student's idea if the teacher and the other students can understand the idea. Although the teacher encourages other students to think about how quarters might be of use in determining three-eighths, the problem cannot be truly shared if individuals hold different concept images for quarters.

REFINING THE LESSON

Although the lesson was generally quite effective at encouraging students' justification of fraction units and in preparing to make a transition from embodied fractions (parts of a whole) to recognising the equal whole needed for comparison of fractions as mathematical objects, revisions are needed to better address two areas. The first area relates to the number of students who reached the desired goal of the lesson. About one-quarter of the class had difficulty generalising the process of repeated halving to create eighths of different sized units. This suggests that it might be helpful to carry out the process of finding three-eighths of a different length, estimating and then using repeated halving, with students clarifying the process before moving to the worksheet. The second area, which is integrally related, is developing *taken as shared meanings* for one-quarter and one-half. Discussion of the strategies used to partition intervals into eighths relied on shared meanings for quarters that did not appear to exist within the class. The next lesson in this unit looked at students partitioning pieces of string using small pegs to construct related fractions. This activity (related to eighths, quarters and halves) may have helped in creating the *taken as shared meanings* for these fractions before discussion arising from the worksheet activity (Appendix B).

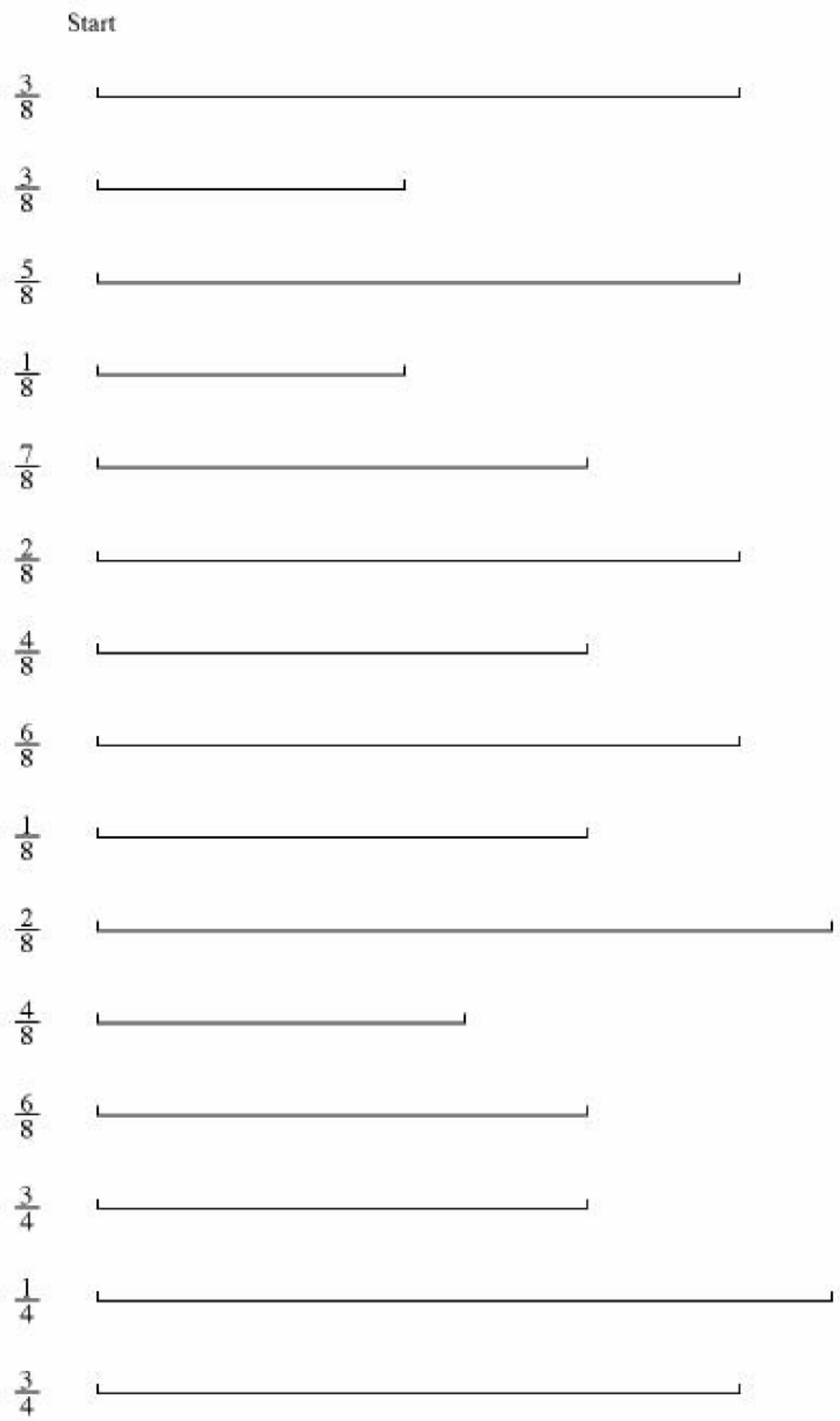
APPENDIX A: ESTIMATING 3/8 OF THE WIDTH OF THE BOARD (GRADE 3-4)

Goal: The fraction 3/8 is used with the attribute of length to focus on composition of partitioning through repeated halving, to link to students' concept images of one-half and one-quarter.

	OUTLINE	**COMMENTS**
Problem posing	Challenge the students to estimate 3/8 of the distance from the left side of the board. Invite three students to mark the distance and to add their initials to the marks they make. T: *How could we determine which mark is closest?* S1: *Measure the distance, divide by eight and multiply by three.* S2: *Find half way, then find half of that to get a quarter, then find half of the distance between one-half and one-quarter.* S3: *Get a piece of string the length of the board and then fold it into eight.*	If the students have difficulty with 3/8 change the question to estimating 1/8 first. Repeated halving is the method used by most students. 
Justifying solution methods	Discuss the strategies. Using a piece of string, invite a student to demonstrate how to check which estimate is closest and justify his or her reasoning. Compare the location of 3/8 and half way. Determine the location of the three crosses in terms of eighths. Use a worksheet involving a number of intervals of different lengths that require students to locate fractions relating to different numbers of eighths (e.g. 3/8, 5/8, 7/8, 1/8) NS2.4 The first two questions ask students to estimate the location of three-eighths on different sized intervals. Check to see if students are locating three-eighths at the same distance from the left, and discuss. *Should the answers be in the same place?*	 Check understanding of the number of folds and the number of parts. 
Comparing units	Put the following question on the board. *Could 4/8 ever be less than 2/8?* Use think-pair-share or a quick feedback back method (e.g. thumbs up, down, sideways) to determine student responses.	This question is designed to prompt the introduction of the need for the equal whole in the transition from partitioned fractions (fractions of things) to quantity fractions (fractions as numbers).

Appendix B

Estimating fractions



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INCORPORATING MATHEMATICAL THINKING IN ADDITION AND SUBTRACTION OF FRACTION: REAL ISSUES AND CHALLENGES

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Since the introduction of the new primary mathematics curriculum in Brunei starting January2006, more teachers of elementary schools are seeking a suitable way of incorporating mathematical thinking in each mathematics lesson that they teach. For this particular purpose, teachers were introduced to lesson study and it is hoped that with the guidance and support given by the team involved, teachers would build their confidence and make mathematical thinking a regular feature of their lessons. There are still many problems that the teachers face such as students not used to explaining their thoughts in class and some insisting on using certain procedures that they had learned before, without being able to explain how and why the procedure works. This paper will relate a classroom case and look at the real issues and challenges that Bruneian teachers faced in incorporating mathematical thinking when teaching of the topic on addition and subtraction of fractions.

Introduction

Mathematical thinking is the mathematical mode of thought that we use to solve any problem in our daily life including at schools (Khalid, 2006). It can be defined as applying mathematical techniques, concepts and processes, either explicitly or implicitly, in the solution of problems (Khalid, 2006). It is, according to Katagiri (2006), the most important ability that mathematics courses need to instill because it makes students able to think and make independent judgement. He also said that mathematical thinking allows for an understanding of the necessity of using knowledge and skills as well as learning how to learn by oneself, and the attainment of the abilities required for independent learning. Stacey (2006) reiterated this fact by saying that mathematical thinking is important because it is an important goal of schooling; it is important for teaching mathematics; and it is an important way of learning mathematics. In fact, the framework used by PISA shows that mathematical literacy involves many components of mathematical thinking, including reasoning, modelling and making connections between ideas. It is therefore imperative that mathematical thinking be stressed in any school curriculum and this is reflected in the new curriculum for primary mathematics of Brunei Darussalam, which was put into implementation from early 2006 (Khalid, 2006).

Children are encouraged to use thinking skills and problem solving strategies during mathematics lessons and not just learn mathematical skills and concepts from listening to the teachers. It is feared that if mathematical thinking is not emphasized, our children would end up learning mathematics by rote memorization, without understanding and without the ability to think intelligently.

More on Mathematical Thinking

The new Bruneian "Mathematics Syllabus for Lower and Upper Primary Schools" (CDD, 2006a; 2006b) considers mathematical thinking as among the processes, skills and values that need to be developed through the teaching and learning of mathematical content. This bears similarity to how Professor Katagiri (2006) defined mathematical thinking. According to him, mathematical thinking can be divided into three categories:

I. Mathematical Attitudes
II. Mathematical Thinking Related to Mathematical Methods
III. Mathematical Thinking Related to Mathematical Contents

The first category is considered as the driving force behind the two latter categories.

"Mathematical attitudes" is a very important affective factor in determining students' behavior in mathematical thinking and problem solving because students' attempts in mathematical thinking depend on how interested they are in problem solving or the lesson. Students' expectation that mathematics will be useful (which involve beliefs) and their personal attributes such as confidence, persistence and organization are mentioned by Stacey (2006) as some of the skills and abilities required for problem solving. Attitudes and values are also mentioned in the Brunei curriculum document (CDD, 2006a; 2006b).

"Mathematical thinking related to mathematical methods" was listed in detail by Katagiri (2006) as consisting of inductive thinking, analogical thinking, deductive thinking, integrative thinking, developmental thinking, abstract thinking, thinking that simplifies, thinking that generalizes, thinking that specializes, thinking that symbolizes and thinking expressed with numbers, quantifiers and figures. Stacey, (2006) quoting from Mason, Burton and Stacey (1982) defined this category as mathematical process that is made up of:

- specializing – trying special cases, looking at examples
- generalizing – looking for patterns and relationships
- conjecturing – predicting relationships and results
- convincing – finding and communicating reasons why something is true

The resemblance of this category in the Brunei curriculum document (CDD, 2006a, 2006b) would be the processes which include mathematical thinking and communication.

"Mathematical thinking related to mathematical content" include ideas of sets, units, expressions, operations, algorithms, approximation, fundamental properties and formulas. These can be compared to mathematical skills (as well as estimation and mental computation) in the Brunei curriculum or deep mathematical knowledge as stated in Stacey's requirement for problem solving.

In the Bruneian syllabus, mathematical thinking and problem solving are mentioned together. Teachers must encourage children to use thinking skills and problem solving strategies during mathematics lessons (CDD, 2006b, p. 7). Among the sub-processes of the mathematical thinking and problem solving processes that are listed in the syllabus are: guessing and checking, drawing diagrams, making lists, looking for patterns, working backwards, classifying, identifying attributes, sequencing, generalising, verifying, visualising, substituting, re-arranging, putting observation into words, making predictions as well as simplifying the problem and solving part of problems.

The curriculum recommends the use of a variety of representations to facilitate the development of the content knowledge and processes. Active learning is encouraged and the use of different representations is to be implemented according to the age and stages of the pupils. In the early years, concrete materials are supposed to help children develop basic mathematical concepts. As children move on, diagrams, real-world examples, verbal representations, ICT and symbolic representation will help children proceed from the concrete to more abstract ways of thinking. The use of symbolism to shift from process to concept is what Tall (2006) termed as ‘procept’.

The Lesson Study Group

Lesson study was recommended by Khalid (2006) as a professional development for teachers to familiarize themselves with incorporating mathematical thinking in their lessons in Brunei. Since many Bruneiean teachers are not very familiar with lesson study, a long-term strategy in the form of research project was developed to introduce lesson study, in order to make it a regular feature in teachers’ professional development and training. A team was established to ensure the smooth running of the project. The team comprises of me (as the project leader), Md. Khairul Amilin Hj Tengah as well as Dr. Hjh. Zaitun Hj. Taha from Universiti Brunei Darussalam, Mr. Palanisamy Veloo from the Curriculum Development Department, and Mr Masaki Takahashi from Sultan Saiful Rijal Technical College. We managed to secure a research grant from the university to help us finance this research study which was approved in June. We managed to attract four teachers attached at a particular Secondary School to join our project and we started the training way before the research grant was approved to make sure they have enough time to consolidate the ideas behind lesson study before we start with actual classes. During training, the philosophy and process of lesson study were explained and with the help of the lesson study videos that were received from Japan, the teachers could clearly see what was meant.

We started our actual lesson around mid April and since the research grant was not yet approved at that time, we manage to record the lessons with a DVD camera belonging to one of us, without even a tripod. When we were about to start the actual teaching process, two teachers withdrew. Of the remaining two, only one attended the meeting regularly to discuss her lesson plans with us. We managed to record the lessons of both teachers teaching different topics. However, for the purpose of this presentation, we will only look at the lessons of one particular teacher.

The Study

The study involved two secondary one (about grade/year 7) classes, 1A and 1E. The two classes comprised 34 and 38 pupils respectively. The main aim of the lesson is to incorporate mathematical thinking into teaching and learning of addition and subtraction of fractions. There are other aims stated in the teacher's lesson plan and they are as follows:

At the end of the lesson, students should be able to:

I. Perform the addition and subtraction of fraction involving:
   - Fractions with like denominators
   - Fractions with unlike denominators
   - Improper fractions and mixed number

II. Solve problems related to addition and subtraction of fraction

Lesson Plan Development

The lesson plan for the purpose of lesson study was written by the teacher after discussion with the team. She was told that since the purpose of the lesson is to incorporate mathematical thinking, she also needs to think of the kinds of questions to ask the student and what to expect from students' responses. She was determined to make students participate during the lesson because communication is necessary for developing mathematical reasoning. The lesson plan can be seen in Appendix A.

For introduction, the problem that she posed for the students was composed to make students interested in the lesson and to stimulate their thinking. This is considered an important part of the lesson because it dealt with mathematical attitude. It is one of the ways to motivate students, as was mentioned in Keller's (1983) ARCS model where students' attention is gained and maintained by innovative posing of problem. The teacher adapted the names of famous international stars and weird long names for this purpose. The teacher also tried to cover the syllabus to include like, unlike and improper fractions in her introductory problem.

Next, the teacher prepared teaching aids in order to help students visualize the problems in a concrete way which can be classified as thinking expressed with figures or manipulatives. Her paper folding activity is also an attempt to make students translate thinking that symbolizes to thinking expressed with figures. The ability of students to translate parts of a round pizza to rectangular parts also involved mathematical thinking because it requires the ability of students to simplify and translate the problem to another equivalent form. She also tried to make children think by generalizing when she asked students to look at patterns (or what happen) when the questions were changed to fractions with larger denominator. At the same time, she planned for students to conjecture about the result when this happens. The learning processes in conjecturing include defining, exploring and constructing premise/conclusion according to Fou-Lai Lin (2006) or predicting relationships and results according to Stacey (2006).

To further assess students' understanding of the lesson, the teacher prepared different sets problems involving fraction magic-squares. Number puzzles and tricks are excellent for featuring mathematical thinking prominently in lesson (Stacey, 2006). Students were made to work in pairs and each pair was given a different set of magic-square. Working in pairs requires ability such as communication and interpersonal skills. The lesson was planned well and the team was eager to observe the lesson.

Lesson Observation and Comments

Below are the comments on the implementation of the lesson based on the observation data that was collected. I will try to identify the elements of mathematical thinking that are present during the lesson.

The teacher attempted to present the lesson via problem-solving strategy. When the problem was posed in the context described, the element of mathematical attitude (willingness to attempt and attempting to discover mathematical problems in daily life) was present. Students were observed to be very excited and can be heard repeating some of the weird and famous names. The teacher was therefore successful in making students interested in the lesson as well as stimulated their thinking. Later, she pasted the teaching aids in the form of paper pizzas on the white board. Interest, enthusiasm and attitude are important to arouse curiosity and she has done that successfully. She then proceeded by asking the students to give her the mathematical statements of the problems. Students had done fractions before and they can therefore build on experiences that were *met-before* (Tall, 2006). The children responded by giving chorus answers. Here, ideas are compressed into thinkable concepts using language and symbolism (Tall, 2006). At this stage, the teacher had used four of the representations suggested by the curriculum – real-life, diagram, verbal and symbolic.

She then proceeded to use the concrete representation (manipulatives), the paper-folding activity. Therefore the only representation that she did not use was the ICT, which I think would be one of the best representations for better understanding of addition and subtraction of fraction. She could have brought the students to the audiovisual room for ICT representation since the facilities there would allow the use of ICT. However, the use of concrete representation involved some amount of mathematical thinking when students need to transfer other representations to this representation. Since the shape of the paper used is not the same as the one in the diagram, children need idea of units (mathematical content) and focus on the constituent elements and their size and relationship (Katagiri, 2006). Children were encouraged to fold the rectangular papers guided by their teacher to understand how the answers were obtained. In the case of unlike fractions, most students prefer the use of lowest common multiple (LCM), which is the procedural knowledge that they had acquired before. Students do not really understand the idea of LCM since even for simple LCM of 4 and 8, they needed to use the algorithm that they learned before, and did not use logical thinking at all. The teacher however, did not take this opportunity to explain LCM in terms of finding equivalent fractions. This is a common feature of a mathematics lesson in Brunei, where teachers teach procedures and algorithm without explanation and students learnt them without understanding. So they were able to do mathematics without understanding what the lesson is all about. There are other instances where rules were given and children try to remember them by heart. When they are many rules to remember, they would be confused and make mistakes.

Students were able to generalize and conjecture (mathematical method), when the teacher guide them to look at the patterns and when fractions with large denominators were used. Since folding paper would be out of the question, this is considered as an important part of the lesson. However, it would have been better if she asked one of the children to communicate their reasoning by asking appropriate questions instead of just getting chorus answers. Most of the explanations given by students were not very convincing.

Before the end of the lesson, children were provided with magic-squares where they have to fill-in the empty boxes. Children were observed to communicate with each other and the teacher to discuss the problem. Although clear instructions were given to the students, almost all of them could not complete the problem.

During discussion with the team, the teacher was advised to teach addition and subtraction of fractions separately, because she had to rush through the lesson due to limited time. The teacher responded that her lesson was planned that way because the students had learned about addition and subtraction of fractions before. However, she will separate them into two lessons for the other class that she would teach that week since it seemed that students still need time to understand about fractions. Our comments were taken positively and we could see some improvements when she taught subtraction of fraction to class 1E. Here we could see that by teaching addition and subtraction separately, the phase of the lesson in 1E was just right. Furthermore, these students are supposed to be a weaker group than the previous one. We were not able to observe the teaching of addition of fraction for class 1E because many of the team members had other commitments on that day.

Discussion

So, what are the real issues and challenges related to the incorporation of mathematical thinking in each lesson in Brunei Darussalam? In my last paper (Khalid, 2006), I have mentioned three perceived issues and challenges:

1. The over-emphasis on examination and examination results.
2. Teachers readiness to teach students to think mathematically
3. Changing the expectations of the stake-holders.

I will however add another one and that is lack of students' participation during lesson.

The first issue is still the biggest issue to be addressed. The children involved in this study were children who sat for their "Primary Certificate Examination" the year before. Considering they have gone through fraction with their teachers during primary levels, one would expect a solid understanding of the topic from them. However, looking at the way they tried to solve the magic-square, it seemed like they were learning the topic the first time. Their persistence on using LCM to solve addition and subtraction of unlike fractions proved the case that they remember algorithm and do not actually understand what LCM stands for. This agrees with the findings of Lim (2000) and Clements (2002) that said that Bruneian children have low level of conceptual understanding of mathematical ideas involved and rely on procedural approaches or rote memorization. Tall (2006) reiterated the fact that procedures that are not compressed into thinkable concepts may give short-term success in passing tests, but if those procedures are not given a suitable meaning as thinkable concepts (in this case, procepts), then they may make future learning increasingly difficult.

The need to reform assessment and evaluation of school children becomes more crucial. Assessment should vary and should not solely depend on one sit-down examination to determine students' progression. At the elementary level, more performance-based and authentic assessment should be introduced. Traditional testing methods in mathematics have often provided limited measures of student learning, and equally importantly, have proved to be of limited value for guiding student learning. These methods are often inconsistent with the increasing emphasis being placed on the ability of students to think analytically, to understand and communicate, or to connect different aspects of knowledge in mathematics. I am however pleased to hear that the ministry of education is doing something about this. As a first step, they could at least set the examination questions into those that need mathematical thinking to solve.

The second issue concerning teachers' readiness to teach students to think mathematically is really an important issue. Teachers are not used to teach this way and have not been exposed to this kind of teaching. Although the idea and method was taught to them during teacher training, they tend to go back to the traditional method once they were posted to schools. I guess old habits die hard. In the attempt to expose teachers to mathematical thinking through lesson study, I have not had the opportunity to extend this to other schools, due to lack of time and resources since the grant for the purpose of lesson study was only approved in June. We also need committed teachers who are interested in lesson study to make it a success. We went through a setback when teachers decided to withdraw from the study. I guess the school should introduce a reward system for teachers who participate in lesson study or other professional development courses. However, I am still hopeful that it will succeed when I hear about the success of lesson study in other countries.

I consider the third and last issue as the most difficult to change. Stake-holders like parents and children should be made to realize that putting too much emphasis on examination results will in the long run lead to the children being disadvantaged. There should be an awareness that the nature of work has changed. Employers nowadays seek workers who are team-players, thinkers and problem solvers. Mathematical thinking provides a solid foundation to produce good problem-solvers. School administrators should not put too much pressure on the teachers to produce good result, because there is a tendency for teachers to take a short-cut to achieve this. Teachers become pre-occupied with preparing students for examinations (Majeed, Aldridge& Fraser, 2001) and students would come to regard mathematical "understanding" being the same as being able to answer examinations questions correctly (Clements, 2002). If we do not change, then the level of scholastic ability of the students will be among the top of the list produced by Katagiri (2006) in the hierarchy of scholastic abilities and mathematical thinking (from lower to higher) reproduced below:

1. The ability to memorize methods of formal calculation and to carry out these calculation
2. The ability to understand the rules of calculation and how to carry out formal calculation
3. The ability to understand the meaning of each operation, to decide which operations to use based on this understanding, and to solve simple problems
4. The ability to form problems by changing conditions or abstracting situations
5. The ability to creatively make problems and solve them

The higher the level, the more important it is to cultivate independent thinking in individuals. To this end, mathematical thinking is becoming even more and more necessary.

Students' reluctance be active participants in class can be corrected through constant encouragement from teachers. There are many reasons for them to be quiet in class, such as being afraid of making mistakes and not being fluent enough in English. However, classroom culture can change if teachers insist and encourage certain behaviors and I am glad to see that more and more of our teachers encouraging students to speak and not being afraid of making mistakes.

Conclusion

Mathematical thinking has been proven to be important and is therefore emphasized in the new Bruneian "Mathematics Syllabus for Lower and Upper Primary Schools". The success for implementing mathematical thinking needs concerted effort from everyone involved. The country needs a well educated workforce with the ability to think and analyze, using varied reasoning and problem-solving skills in an integrated manner for national development. In order to be able to independently solve problems and expand upon problems and solving methods, the ability to use **"**mathematical thinking**"** is considered even more important than knowledge and skill, because it enables to drive the necessary knowledge and skill (Katagiri, 2006).

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APPENDIX A

Lesson Plan of the First Lesson Observed

Name of Teacher: Hajah Nuzailah
Class: 1A (Year/Grade 7)
Topic: Addition and Subtraction of Fraction.

Learning Objective: So that children are able to think mathematically how to solve problems related to addition and subtraction of fractions (like and unlike fractions).

Other Objectives: 1) For students to be interested in the lesson (have the right attitude to involve in mathematical thinking)
2) For students to think mathematically in integrating what they know and relate it with what they are currently learning and to make sense of the lesson using the manipulatives provided.

Introduction: Pizza problem

These students ordered some pizzas from the pizza vendor and ate them according to the following proportion:

Name	Amount of pizza
1. John Beckham	1/3
2. Ramziah Khuzaifah	1/3
3. Shawn Wayne	1/5
4. Agus Muslimah Qawiyah	3/8
5. Hendrick Schumacher	1½

Questions Posed	**Corresponding thinking**
1. What is the amount of pizza John and Ramziah have together?	Students are supposed to add like fractions
2. If John and Ramziah gave up 1/3 of their total share, what is the amount they have left?	Students are supposed to get the sum from question1 and subtract ¼ from it. Subtract like fractions
3. What is the amount that both Shawn and Agus have together?	Adding unlike fraction
4. What is the amount if Shawn and Agus give up ¼ of their share to a friend?	Getting the sum from question 3 and subtract ¼ from it
5. What is the amount that Shawn and Hendrick have together?	Adding unlike fractions involving improper fraction
6. Shawn and Hendrick gave up 2/3 of their share. How much do they have left?	Getting the sum from Question 5 and subtract 2/3 from it
7. How many pizzas did they order altogether?	Getting the sum of all the fraction of pizzas And rounding them

Activity: Paper Folding

Students are given blank papers, and are encouraged to translate the questions using he rectangular paper to get answers (by folding and shading). Students were asked to look at patterns and as follow-ups, the problems are extended to larger denominators like $13/20 + 4/20$, $2/15 + 4/12$, $1\frac{7}{8} - 1\frac{1}{3}$

Student Evaluation: Magic Square with fractions

Students were supposed to work in pairs and fill up the empty boxes with fractions that will make each row and column equal to 1. They are not supposed to use the same fraction more than once.

Lesson Plan of Second Lesson Observed

Name of Teacher: Hajah Nuzailah
Class: 1E (Year/Grade 7)
Topic: Subtraction of Fraction.

Learning Objective: So that children are able to think mathematically how to solve problems related to subtraction of fractions (like and unlike fractions).

Other Objectives: 1) For students to be interested in the lesson (have the right attitude to involve in mathematical thinking)
2) For students to think mathematically in integrating what they know and relate it with what they are currently learning and to make sense of the lesson using the manipulatives provided.

Introduction: Is there enough Pizza?

Jamil bought one large pizza and plan to share the pizza with his friend. He divided the pizza as follows:

Name	Amount of pizza
1. Jason	¼
2. Jill	1/8
3. Jamilah	3/8
5. Johari	2/5

Questions Posed	**Corresponding thinking**
8. If Jason is the first to eat the pizza, how much is left for the rest	Students are supposed to subtract a quarter from 1 to get three quarters.
9. If Jamil gave 1/8 of the remainder to Jill, what fraction is left?	Students are supposed to subtract 1/8 from three quarters
10. During break time, Jamilah ate 3/8 of the remaining pizza. How much pizza	Subtracting like fraction

is there left?	
11. Is there enough pizza for Johari? Is his share bigger or smaller than Jamilah?	To test students' understanding on order of fraction (which is bigger and smaller
12. What happen to Jamil's share? Is it getting more or less?	To make student think what happen during subtraction

Activity (Paper folding)

Students are given blank papers, and are encouraged to translate the questions using he rectangular paper to get answers (by folding and shading). Students were asked to look at patterns and as follow-ups, the problems are extended to larger denominators like $^{13}/_{22} - ^{4}/_{22}$, $^{4}/_{15} - ^{2}/_{12}$, $1\frac{7}{8} - 1\frac{1}{3}$. Although there was no question involving mixed number or improper fractions above, children were asked one question involving these fractions during the activity.

Student Evaluation: Worksheet on subtraction of fraction

The questions on this worksheet were the normal exercise book problems that we often see.

Appendix B (Video Description)

Title: Addition and Subtraction of Fraction

Teacher: Hajah Nuzailah Haji Nali

Class: Form 1A and 1E (Grade 7)

School: Sekolah Menengah Masin

Date: 1st lesson – 14th April, 2007 for class 1A
2nd lesson – 24th April, 2007 for class 1E

Team member: Dr Madihah Khalid (UBD), Mohd. Khairul Amilin Haji Tengah (UBD), Dr Zaitun Hj Taha (UBD), Mr Masaaki Takahashi (MTSSR), Mr. Palanisamy Veloo (CDD)

Introduction

The first lesson was on the topic of addition and subtraction of fractions. The teacher planned the lesson very well, laboriously prepared the teaching aids, determined fully the procedures of her teaching and prepared interesting activities. Everything was written in her lesson plan as shown in Appendix A. Since the aim of the lesson was to incorporate mathematical thinking into each lesson, she has also prepared to ask appropriate questions to students directly to elicit students reasoning.

In the introduction of the lesson, the teacher tried to motivate the students by telling a story so that children would be interested to learn and be involved in class. This is one aspect of mathematical thinking, (the mathematical attitude) that the teacher is trying to address. The names of the characters in the story were adapted from the names of famous people. She probes and asks students questions but usually gets chorus answer. She could have improved her questioning techniques if she set some rules like whoever wants to speak in class should raise their hands-up and she would in turn call out the children. From our discussion, we think that she should have asked the students why “when it comes to addition and subtraction, the denominator need to be the same”.

Activity (Paper folding)

Paper folding was used to help students translate numbers and symbols to the concrete form for better understanding. Transferring round pizzas into rectangular papers also needs imagination and is also another aspect of mathematical thinking. It is also here that the teacher should stress the importance of equal parts, to allow these parts to be added or subtracted together. Maybe, ask one of the students to explain this.

Some students are not really interested in folding papers since they were already taught the addition/subtraction algorithm. When faced with addition or subtraction of unlike fractions, they would always suggest finding the “LCM” or “least common multiple”. I found that children don’t really know what LCM really is. They just know how to find them. Even when asked “What is the LCM of 2 and 4, they insist on performing the division” (algorithm).

Children are still not communicative enough and this is where a teacher’s skill in probing would be handy. In my opinion, students in Brunei are still not participative enough in the class and this lack of communication and the inability for them to explain their thoughts make it a challenge for teachers in Brunei in reading their thoughts (mathematical thinking).

Assessing students’ understanding

Teachers are supposed to encourage students’ self evaluation by asking right questions. However, only a few pupils could explain well. Therefore, to further assess their understanding, students were asked to work in pair to work out the answers to the magic square with fractions. Different pairs of students were given different number combination. An example of one of the combination given is as follows:

	$^{10}/_{20}$	$^{3}/_{9}$
$^{4}/_{12}$		
		$^{6}/_{16}$

Children were found to still struggle to complete this simple exercise and they could not finish it in class because time was up. They were however encouraged to complete it at home and bring to the next lesson. During discussion with the teacher, she was advised to just concentrate on addition in one lesson and subtraction in another lesson, because one hour is too short for both.

The Second Lesson

The second lesson that the team observed was a lesson subtraction of fraction with class 1E. These pupils had a lesson on addition of fraction five days before and the team could not observe the teacher.

Evaluation of the lesson

Again, to make the lesson interesting, names of students in the class were used. It aroused students’ interest and stimulated their mathematical thinking. This lesson was better executed than the previous one and children enjoy it because they have enough time to think and do the activity with their teacher. However, there is still the tendency for pupils to solve problems using LCM. Although the teacher tried to make them think in terms of equivalent fractions, they still insist on solving it using LCM. I guess old habit dies hard.

LESSON STUDY AS A STRATEGY FOR CULTIVATING MATHEMATICAL TEACHING SKILLS
A CHILEAN EXPERIENCE FOCUSED ON MATHEMATICAL THINKING

Francisco Cerda B.
Ministry of Education, Chile

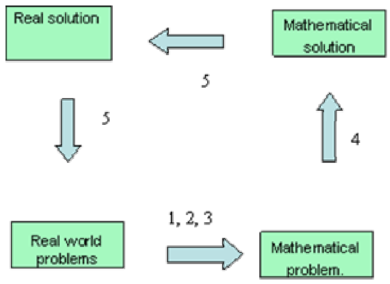
Introduction

In Chile, approximately 100% of school age children are registered in primary school. Nevertheless, the school attendance does not ensure that mathematics learning is of the expected quality. The testing results indicate this lack of quality in math learning. There is great interest in our society as a whole to focus on a qualitatively change in the practice of math teachers. Chile has been carrying out multiple initiatives in this sense for several years and lately it is participating in a project to improve the education of mathematics with the technical assistance of Japan. At the same time, within the framework of the collaboration between the economies of APEC, Chile participates in a collaborative project in mathematical education that contemplates a progressive study of different subjects related to mathematical education: good practices, mathematical thinking, communication, evaluation, and generalization towards other subjects. This report gives an account of an experience that, following the methodology of Lesson Study, was developed according to the didactic principles of Consultant's School Strategy for the Curricular Implementation in Mathematics, strategy developed in Chile. The comparative analysis of the two previous strategies, made by Gálvez (2006), *allows us to conclude that both constitute powerful strategies to improve the teacher practice and, at the same time, to generate processes of professional learning of the teachers, which guarantees a greater stability in the changes obtained by its performance*.

Lesson Study focused in the mathematical thinking.

To focus the processes of study in the development of the mathematical thinking implies a double challenge; it involves the students as much as math teachers. Professor Katagiri (2004) aiming at the autonomy of the students indicates: "*Cultivating the power to think independently will be the most important goal in education from now on, and in the case of arithmetic and mathematical courses, mathematical thinking will be the most central ability required for independent thinking*".
On the other hand Stacey, K. (2006) emphasizes that "*providing opportunities for students to learn about mathematical thinking requires considerable mathematical thinking on the part of teachers*"

A key window considered in Chile to develop the mathematical thinking mentions the mathematization of real world phenomena. The cycle of mathematization has been described by means of the following scheme: (Jan de Lange, 2006):



1. Problem situated in reality
2. Identification of the relevant mathematics
3. Gradually trimming away the reality
4. Solving the mathematical problem
5. Interpret the meaning of the mathematical solution in terms of real world.

To connect a problem of the real world with mathematics is not trivial and requires a fundamental mathematical competence. According to (OECD-PISA), today at least eight key mathematical competences are distinguished, that allow this connection; they are: to think and to reason, to argue, to communicate, to model, to create and to solve problems, to represent, to use language and symbolic operations, formal and technical and, to use aids and tools.

The Consultant's School Strategy for the Curricular Implementation in Mathematics (LEM), sustained by the Ministry of Education in alliance with Universities of the country, is based on the following didactic principles for the learning of mathematics (Espinoza, 2004)

1. In order to learn, the student must take part significantly in the mathematical activity, and not only limit him/herself into accepting and applying the strategies taught or “shown” by the teacher.
2. Learning consists of a change of stable strategy, the replacement of knowledge by another one, caused by an adaptation to a situation.
3. The mathematical knowledge arises from the work of the children as the optimal answer to specific problematic situations that require it.
4. The learning activities must constitute true challenges for the children, when putting in crisis/conflicts their previous knowledge. These activities must be accessible to the children and have their frame of reference in familiar and significant contexts.
5. When the teacher (or the text) gives the necessary instructions to do the task correctly, it is the teacher who is using the required mathematical knowledge and not students.
6. The mathematical knowledge in a learning process must appear as the necessary knowledge to pass from the initial strategies, low efficient and inadequate, to the optimal one.
7. The students choose and share different resolution techniques. The “error” is a substantial part of the learning process.
8. The knowledge and mathematical procedures used must be valued by all the children. It is recommendable not to spend a long time between the moment at which the mathematical knowledge has emerged for the children, and the moment

at which the teacher emphasizes and systematizes it. But, it does not have to be formalized prematurely.

9. The students must have the opportunity to work and to deepen the knowledge until obtaining a significant dominion of it.
10. The argumentation and mathematical explanation are seen as laying the basis for the adjustment of the algorithms and the modification of the mistakes.

Description of a Lesson Study experience in Chile focused on Mathematical Thinking

This study took place in the Municipal School *Dr. Luis Calvo M.* (Santiago), but it had the participation of other teachers as observers, they belonged to two other schools (*R.I. School* and M, A.C. School), and they wished to become future developers of the designed lesson. The criterions to choose the school were, they have at least, one course per level, to have the conditions to develop an authentic experience, not to simulation, and the schools belong to Whole School Pilot Project (supported by the Ministry of Education and constituting part of Consultant's School Strategy for the Curricular Implementation in Mathematics). The process that is described in this work contemplated six phases that are described briefly next:

Phase 1 Presentation of Lesson Study Project to the group of invited teachers.

The first conversation was carried out in the school Luis Calvo M. and was attended by the mathematical teachers of that school plus the invited ones. They observed a PPT with the foundations of the Collaborative APEC Project in Chile, whose long term goal is to promote the development of mathematical thinking from the perspective of the mathematization processes. The author of this report assumed the role of external adviser, summoning the group and carrying out the presentation of the project. They discussed the following text (Takahashi, A., 2006): *the idea of lesson study is simple: collaboration with fellow teachers to plan, observe and reflect on lessons developing a lesson study, however is a more complex process Because lesson study is a cultural activity, an ideal way to learn about lesson study is to experience it as a research lesson participant. In so doing, you will learn such things as how a lesson plan for lesson study is different from a lesson plan that you are familiar with, why such detailed lesson plan is needed, what type of data experienced lesson study participants collect, and what issues are discussed during a post lesson discussion*.

One of the teachers stated that this group had elaborated, last year, a Didactic Unit (from now on DU) of Geometry in a course of teacher training and that had been only applied by one teacher, in October, 2006, but the other designers could not observe it. This DU was directed to 8th grade; its subject was the area and perimeter of circumferences and consisted of at least eight classes. Since the subject of geometry is of particular interest, they agreed to start off with a critical revision of this DU before carrying out a new design. They agreed, in addition, to invite the teacher who had applied the Didactic Unit with the purpose of knowing first hand, how was this process. He was given a task of reading and of studying that Didactic Unit. Eight sessions were set up as a minimum to develop this project. All the assistants valued the opportunity to meet and reflect on the education of mathematics. They said that they had the idea to form a reflection group on mathematical education, that they would call *Pythagoras Group.*

Phase 2 Critical revisions of the subject area and perimeter of a circle
Teacher C.M, who worked on this subject, told them how he lived the experience when he applied the didactic unit designed by them. In particular, the discussion focused towards the second class that corresponds to the characterization of the number π. The class contemplated three moments:

- Measurement of the contour of circular base objects, using a ruler or measuring tape.
- Searching for a broad approach of how many times the diameter fits in the circumference (arriving at that it is 3 times and "something more". The lacking difference is not quantified.
- from measurements of the contour **P** of a circle and its diameter **d**, calculates the reason: **P/d**

The following photos illustrate the moment at which the students find that the diameter fits at least three times in the contour of the circumference. In doing this they bordered with plastilina the contour of the circumference.

There was an intense and interesting discussion that was centered in the following key points:

- Why the students did not quantify how 3 diameters need to cover the perimeter of the circumference? This course already has the tools to have done it (but, they only obtained value three).
- The plastilina had a problematic performance because it is tensile. Some children stretched it to fit it into the circle contour three times.
- When they did the quotient between the measures of the perimeter and diameter of the circumference (cm.), obtained numbers with three or more decimal, but those decimals cannot be considered as valid values; there exist a physical limitation in the measurement that prevents obtaining more precision than mm

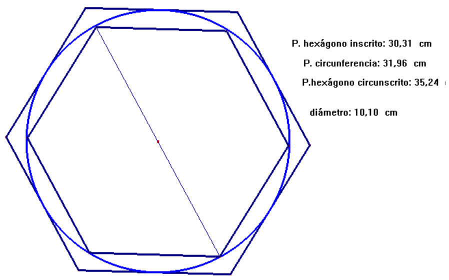
gives. In spite of this, the students obtained decimal values for the quotient, like the following ones.

	Perimeter cm	Diameter cm	Perimeter/diameter
Circle 1	25,1	8	3,1375
Circle 2	19	6	3,167
Circle 3	12,7	4	3,175

- The Teachers' team decided to focus their lesson design in the measuring of the contour of the circumference using as a unit of measurement its diameter and **quantifying** the magnitude of the remaining segment (smaller than a diameter).
- Another question that arose: which is the better moment for presenting the number π? Since it is very common in present education that π is defined early as the quotient **P/d.** They thought it was not necessary a premature definition of this number, it is better to obtain first an approximate experimental value and then to characterize it in a stricter form.
- One of the teachers was requested that he simulate a class about this same subject matter, but under the traditional way. He did it, and then an interesting discussion took place around the differences in the mathematical thinking that is put into play in each modality. The first proposal of a class was sketched and a task was given: advancing in the design of the class in individual form.

Phase 3. Design the research-class.

The design of the research-class begins. It will center in obtaining a quantitative relation between the perimeter of a circle and its diameter. For this purpose they will set out a sequence of measurements of the contour of a disc, varying in each case the measurement unit that is used: initially without measurement instruments, then using a graduated ruler, continuing with a metric measuring tape, to culminate with the use of the diameter as measurement unit. Product of the previous process they will obtain a good approach for the number π.



Teacher A.F. raises the idea to complement later the work later of this class by means of an activity with the geometric software Cabri Géométre II that will consist in drawing a circumference with a inscribed and circumscribed hexagon in it, getting the respective measurements of the perimeters with the Cabri tool for measure, and comparing those values with the experimental values obtained in the designed class. Also he raises the idea that the students analyze a documental about the interesting history of the number π. There remains a task: to write up a draft of the class plan.

Phase 4. To complete the lesson, establishing the necessary materials.

The intentions of this session are: to raise the hypothesis about what will happen in the planned class, to prepare good questions for the students, to anticipate possible ways of solving the problems, to raise and to assign different tasks about the observation of the participants. They agree that the students will work in small groups and will have discs on which they will carry out the measurements. Each professor brought different circular objects. They decided on covers of round plastic containers. The materials and the activities are proven. When carrying out the quantification of how many times the diameter d fits in the perimeter P of the circumference, they obtained an approximated value of $3\frac{1}{9}$, of the reason P/d. The used technique consisted of the segment that exceeds 3d was marked on the paper tape (of length d) and then this one was bent successively. They checked that the segment fitted 9 times. They defined a sequence of two classes: In the first class the contour of a disc will be quantified, but in the course of the lesson the conditions of that quantification will change in a progressive form: they measure first with any no conventional unit, then with a ruler, later with a metric tape and it will be culminated with a **measurement that uses the diameter** as the unit of measurement. One hopes that the students quantify the leftover segment after applying 3 times the diameter in the contour of the circular object. That quantification, according to the previous knowledge would have to be expressed through a fraction of the type $P = 3\frac{a}{b}$ times the diameter.

Phase 5 the lesson was taught

The two designed classes were applied in a consecutive form and both were observed by teacher R.L of the same school. In addition the lesson was recorded by the knowledgeable other. This report will be centered in the analysis of class 1 only.

Phase 6 Discussion after the lesson implementation.

The objective of this phase is to put in common the observed things, as well as the performance of the teacher and the answers of the students, to propose ideas to improve the class plan and to make decisions with respect to giving continuity to the joint work. The group of teachers evaluated class 1, through the observation of a video clip of 30 minutes. From that reflection there were the following conclusions:

- In general the class was developed with fluidity and they noticed that the students were involved in it.
- The closing of the class was missing with questions directed to the students, nevertheless the professor made many questions to the groups but not to the entire course.
- With respect to the lived experience, the teacher who led the class indicated: "*I could not always teach classes with this same modality. As an activity for introducing a subject, I agree, I believe that the richness of the learning comes from this, but gradually or at some time it is necessary to return to the role of the "demanding mathematical teacher", because they are going to face new texts, new institutions, other teachers and they have to be prepared*."

Another teacher responded to him: *"We are looking for the participation of the students in the construction of new learning. What happens in the traditional classes is to start off giving the value of π and the formula of the perimeter of the circumference. Here everything is oriented to the origin of π and to find out that the circumference perimeter is independent of the circumference size. If we understand that, we have made a significant advance in the understanding of everything that comes ahead."*

- About the role of the observer: he must have a more active role in the gathering of information. The observer teacher indicated that he felt a desire to interact in the class with the students by asking them some questions. In fact he did it more than once. It was recognized by the group that for a research-lesson like this one, it is very difficult for the teacher of the course to pay attention to everything that happens in the classroom.

In relation to mathematical aspects, they remarked:

- It is necessary to focus on having the students understand that there is a (liner) dependency between the perimeter of the circumference and its diameter. It was stated that there is not a good estimation of which is the measure of the contour of a circumference; in general, people think that it is less than the real measure. A teacher remembered that in certain professional activities they used a metric wheel that in each turn measured 1 meter, and asked the other teachers. Which is the radius of that wheel? And the answer took some time.
- In a another group, when measuring the contour with the diameter, one student said, "it fits 3 times and exceeds a small piece, but we cannot say that it fits 4 times". Another one said: it fits three times and 2 cm. (they mix the units: the diameter and the centimeter).
- With this work they arrived directly to an algebraic expression to find the perimeter of a circumference, it was the consequence of the previous process and it was necessary to formalize it. So far they had obtained a very good approach for π of $3\frac{1}{7}$
- The observer teacher indicated that the teacher missed insisting that in all cases it gives a constant value for the quotient P/d, independent of the size of the disc. *"There is not a little π for small circumferences, and a great π for large circumferences"*
- The teacher responsible for the class said: "I need the students to work with Cabri so that they see that π is a quotient comparison and it has the same value in any circumference", that is to say, π is a ratio. In relation to the question about whether these students have learned the concept of ratio, the answer is no, there is a need to discuss it further in order to see that this ratio is constant. The teacher concludes: he could review the subject about the meaning of π when the subject of proportions will be studied later in the class. It is necessary to arrive to other a problem in which π is used. Another teacher asked: In what other subject π is used? , and the external adviser indicated that in probabilities. There is an experiment of Buffon (century XVIII) that consists of throwing a needle of length

b, on a board divided by lines separated from each other by a certain distance ***a*** (smaller than ***b*** or just as ***a***). Knowing that the probability that the needle falls on one of the lines is 2b/ (a• π), so we can consider π like in the previous case. We can simplify the experiment taking ***a*** = ***b***, with which the probability of the event is 2 /π, and then dividing 2 by the frequency whereupon the event happens we will have our approach of π be the same result as when Buffon made the experiment throwing the needle and obtaining π until with 3 decimal numbers.

- When seeing the techniques that the students used to respond to the challenges raised in the class, we verified that there are some techniques that the teachers could not anticipate: "*When we thought about the class we did not imagine this.*" For example with the use of the ruler: that they were going to draw the circle on the paper and soon they would mark cords of 1 cm, one after the other. In the case of an 18 cm diameter circle, they managed to mark 56 cords of 1 cm, that is to say, they registered a regular polygon of 56 sides. The result has an error of less than 1%; we remembered that with a regular hexagon the error registered is of ≈ 4.5%.

Relative to the performance of the students

- In a group, when a student was asked to explain what he obtained, he refused by saying: "*I cannot express concepts with words*". Teachers reflect on this point asking themselves if it is a language problem. One of them indicated that the student did not know, therefore he cannot explain. Another teacher answer him that he does not think the same, since he observed that the student participated actively in the class, thus he thinks that he lacks mathematical language to express what was experimented. Another teacher commented that one of the key competences in mathematics is communication and that it is neglected in the mathematics class.
- The teacher who led the class indicated that his evaluation of the lesson was positive, he was impressed by the children movement in the room, the spontaneous discussion and the speech of each one of them: "I felt them to be true protagonist of their own learning."

ANALYSIS OF THE LESSON

The following analysis is taken from the Anthropological Theory of the Didactic. (Chevallard, 1997). The subject treated in this class can be considered like an isolated mathematical organization conformed by a mathematical task, the techniques that allow making this task and by a theoretical speech that allows us to explain the techniques and to give a theoretical sustenance to them.

Mathematical task of the class:

- To quantify the perimeter of a circle.

Didactic variables:

- Availability of a circular object (disc) in class 1 or the drawing of a circumference on the paper. (in class 2)
- the type of measuring instrument available to carry out the measurement, or the unit of measurement to be used in the quantification (example: the length of the diameter)

Conditions: these vary progressively throughout the class:

- does not have measurement instrument, only have the disc
- use of one ruler,
- use of a metric tape
- has only a measuring tape of equal length to the disc diameter.

Techniques:

- Uses pieces of paper like a unit of measurement
- Uses its fingers like unit of measurement
- Marks a point of the circular object and makes it roll on to the ruler until the marked point returns to the initial position.
- They copy the contour of the circumference in a sheet of paper and with a ruler they measure with cords of 1cm drawn up one after the other.
- Turn the ruler around the disc. They border the circular object with the paper metric tape measurer

- Put the tape on top of the diameter in the contour concluding that it fits 3 times, but it exceeds a segment smaller than the diameter.
- to quantify the segment that is left successively doubling by half the unit of measurement and obtain the fraction $\frac{1}{8}$ of the diameter as an approximate value (division of the unit)
- Cut a tape to the length of the leftover piece and successively put it on top of the diameter to find out how many times it is possible to fit it (repeat the measure of the piece as many times as it is necessary to cover the diameter completely).

In the development of the class three essential moments can be distinguished:

Beginning Moment: When the mathematical task of this class is presented and it consists of measuring the contour of a circular object. The students react at this moment, according to the conditions put forward by the teacher, without any conventional measures (they measure with the fingers or a piece of paper).

Development Moment: Starting from a progressive change of conditions (availability of ruler, tape, the diameter as unit of measurement) the students elaborate other techniques of resolution. Here the students are faced with diverse obstacles according to the instruments they use, if it is a ruler they have the problem of measuring a curved line with a straight instrument, a thing that is avoided when they use a metric tape, because this one can take the form of the circular object that is being measured. The culminating moment arrives when the students are asked to quantify the contour of the disc only using a piece of paper of equal length to the diameter of the disc. In the first approach, all obtain the result that the diameter fits three times in the contour, but exceeds a small arch that must be quantified when they only have a unit of measurement greater than the length of that arch. Thus, it is a problem, because normally one measures with units smaller than the object to be measured. This implies, necessarily, that the unit of measurement will have to be divided. It is here where diverse techniques arise to obtain the fraction of the diameter. Some students (successively) double the unit of measurement by half until they obtain the eighth, verifying that this fraction of the diameter is very close to the length of the leftover arch. Others mark, on the unit of measurement, the length of that arch and soon they double (successively) according to that measurement, to obtain, either a $\frac{1}{6}$, a $\frac{1}{7}$ or a $\frac{1}{9}$.

Closing Moment: the proposed tasks are reviewed and the techniques used are compared. The new knowledge is identified and institutionalized: the diameter of the circle (measured in cm.) multiplied by some of the following fractional numbers can obtained good approaches to the perimeter of a circumference by multiplying: $3\frac{1}{6}$, $3\ \frac{1}{7}$, $3\frac{1}{8}$, $3\frac{1}{9}$, $\frac{1}{7}$ being the best approach, since it gives the first two decimals in an exact same form: 3,14. The class culminates with a verification activity that consists of: each group must designate a member to go to the blackboard (adherent paper tape exactly to the length of the perimeter of their disc. Then he returns to his group and verifies that it is the right measurement to cover exactly the contour of the disc, without lacking or going over adherent tape. The group that does it well, wins. The class was called: "**How much does a wheel move in one turn?** The students learned to quantify the length of a complete turn of a wheel. From the point of view of modeling (one of the key mathematical competences), we can say that in this real world problem a mathematical structure was imposed: $P = 3\frac{1}{7} \bullet d = \pi \bullet d$, formula that relates the perimeter of the circumference with the length of the diameter.

Final Reflections, Conclusions and Projections

Having finalized this first part of the project we can indicate that it had many benefits for all the participants. The students were committed and enthusiastic with the work proposals; they enjoyed the activities, discussed among themselves and with the teacher the mathematical topic of the class. For the teachers it meant having a longed for space for dialogue and reflection about math education, not only with teachers of the host school, but that with teachers of two other schools. In relation to the institutional aspects, it is necessary to emphasize the strong support of the Director and the Technical Pedagogical Sub-Director of the School Dr Luis Calvo Mackenna, since they gave all the facilities to find the space and the time for meetings of the team that designed and tested this lesson study.. This reminds us of what a director of a North American school stated:

"If we are serious in the fight to constantly elevate the quality of the teachers we must provide to the teachers the time and the resources to them that they need to form a

School	Mathematical teachers who will implement the lesson	8th level courses	Students
Luis Calvo Mackenna	2	2	70
República de Israel	2	2	70
Miguel Ángel Cruchaga	2	3	120
Total	**6**	**7**	**260**

community that invests as much in the learning of the children as of the same teachers. If we are serious in not leaving no child behind, then we must provide a process so that the teachers assume the responsibility of a growth and continuous improvement that does not leave no teacher behind either". (Liptak, L. 2005)

Limitations

- Little time available for the teachers to meet, in order to plan and reflect. They have too many hours of class teaching in their contracts.
- Difficulties to carry out the observer role, for the same previous reasons. To observe the research-lesson means to be absent from their own classes.
- Necessity of a "knowledgeable other". At least during a period of time the intensive support of an external adviser person to the school is necessary. This support should probably go away smoothly until the participants gain experience and autonomy.

Projections

The host school is thinking to apply this research-lesson in another parallel class (8° B). The teachers of the schools that also participated as observers will apply this class in their respective classes; for this purpose a meeting will be held also with schools that did not participate in the project, but which are interested in applying it. The sustenance of this methodology has a good perspective in this school, since there is a supporting principal, depending on the disposition of the teachers to continue practicing it. The idea exists to invite teachers of the neighboring schools to participate in a session in which the class with real students is demonstrated and all the previous ones within the framework of a plan that can soon be discussed with them (LS Open House) to improve the learning of geometry. The following picture shows the quantitative projections of the application of this class:

Finally we wish to indicate that research demonstrates, (Nordenflycht, 2000), that the effectiveness of the changes in the educational practices is related to the level of participation of the involved ones in the process, of the degree of depth of the reflection and the analysis that they carry out in their own practice. On the other hand, the development of a common project requires the rupture of the isolation in which the teacher develops its professional work. This isolation in the professional exercise constitutes, without doubt, a brake for the generation of a collaborative work that is one of the foundations of professional development. Consequently, a proposal to improve has to make it possible for teachers to reflect on their own practice, a collaborative work in which the investigation and the innovation are closely bound in their role as guides and promoters of learning. It implies, in addition, to develop competences and strategies, to analyze and to interpret situations, and to promote viable and effective solutions and alternatives of qualitative improvement. An independent teacher is a subject able to carry out a design on his own, able to interpret his reality and its context, to take initiatives, in synthesis, a constructor of innovations.

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APPENDIX

General Information

Title: How much does a wheel move in one turn?

Topic: Perimeter of a circumference obtained from measuring the contour of a disc with a tape whose length is equal to the length of his diameter

Producer: Ministry of Education, CHILE.

Video recorder and video editor: Francisco Cerda Bonomo

Teacher: Alejandro Flores

Research Team: Rafael León, Alejandro Flores, Soledad Cortes, Christian Méndez

Knowledgeable Other: Francisco Cerda B

Collaborator: Ms. Grecia Gálvez

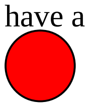
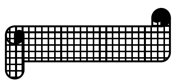
Host School:: Escuela Municipal Dr. Luis Calvo Mackenna. Santiago
Principal: Sr. Patricio Morales Borbal
Pedagogical Coordinator: Ms. Carmen Corvalán Fernández

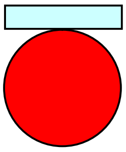
Invited Schools: Escuela Miguel Ángel Cruchaga, Puente Alto. Santiago
Escuela Municipal República de Israel, Santiago

Grade: 8th year of Primary School

Date: June, 2007

1. LESSON PLAN

Lesson 1	**Conditions** (teacher vary progressively throughout the class)	**Techniques** used by the students who allow them to make the mathematical task under specific conditions	**Remark**
Mathematical task to quantify the perimeter of a circumference	▪ They do not have measurement instrument, they only have a plastic disc 	• Use its fingers like unit of measurement • Use a piece of paper like a unit of measurement	Ask to the students: ▪ what value you consider it will have the length of the contour? Write down it! ▪ What difficulties present the measurement with a ruler? ▪ What differences you consider exist between the measurement with a ruler and with a paper tape?
	▪ use a graduated ruler (in cm and mm) 	• Marks a point of the circular object and makes it roll on to the ruler until the marked point returns to the initial position. • They copy the contour of the circumference in a sheet of paper and with a ruler they measure with cords of 1cm drawn up one after the other. • Turn the ruler around the disc.	
	▪ use of a metric tape 	• They surround the disc with a metric tape	

	Using a piece of paper of equal length to the diameter of the disc. Students will be requested to quantify the magnitude of the segment that exceeds 3 times the diameter. (This remaining segment is smaller than a diameter). 	• Put on top of the diameter in the contour concluding that it fits 3 times, but exceeds a segment smaller than the diameter. • to quantify the lacking segment successively doubles by half the unit of measurement and obtains the fraction 1/8 of the diameter like approximate value (division of the unit) • Cut a tape of the length of the leftover piece and successively put on top it in the diameter to find out how many times it fits is possible (to repeat the measure so many times of the piece as many times it is necessary to cover the diameter completely).	What fraction of the diameter is the leftover segment? They will be asked for to cut a piece of tape equal to the contour length of the disc.

2. EXPLANATION OF VIDEO

- **Title of VTR**: *"HOW MUCH DOES A WHEEL MOVE IN ONE TURN"?*

- **Summary**

This video shows a lesson for the eighth course of primary school. The study subject talks about the quantification of the perimeter of a circumference from the measurement of the contour of a circular object. From the conditions that the teacher is putting for the measurement, a progressive elaboration of techniques on the part of the students takes place. The central activity consists of which they measure the perimeter of the circular object using a tape whose length is the measure of the diameter of that object. One hopes that the students quantify that measurement using the whole unit of measurement and a fraction of it

- **Components of the lesson and major events in the class**

In the development of the class three essential moments can be distinguished:

Beginning Moment: When the mathematical task of this class is presented and it consists of measuring the contour of a circular object. The students react at this moment, according to the conditions put forward by the teacher, without any conventional measures (they measure with the fingers or a piece of paper).

Development Moment: Starting from a progressive change of conditions (availability of ruler, tape, the diameter as unit of measurement) the students elaborate other techniques of resolution. Here the students are faced with diverse obstacles according to the instruments they use, if it is a ruler they have the problem of measuring a curved line with a straight instrument, a thing that is avoided when they use a metric tape, because this one can take the form of the circular object that is being measured. The culminating moment arrives when the students are asked to quantify the contour of the disc only using a piece of paper of equal length to the diameter of the disc. In the first approach, all obtain the result that the diameter fits three times in the contour, but exceeds a small arch that must be quantified when they only have a unit of measurement greater than the length of that arch. Thus, it is a problem, because normally one measures with units smaller than the object to be measured. This implies, necessarily, that the unit of measurement will have to be divided. It is here where diverse techniques arise to obtain the fraction of the diameter. Some students (successively) double the unit of measurement by half until they obtain the eighth, verifying that this fraction of the diameter is very close to the length of the leftover arch. Others mark, on the unit of measurement, the length of that arch and soon they double (successively) according to that measurement, to obtain, a $\frac{1}{6}$, a $\frac{1}{7}$ or a $\frac{1}{9}$.

Closing Moment: the proposed tasks are reviewed and the techniques used are compared. The new knowledge is identified and institutionalized: the diameter of the circle (measured in cm.) multiplied by some of the following fractional numbers can obtained good approaches to the perimeter of a circumference by multiplying: 31/6, 31/7, 31/8, 31/9, 31/7 being the best approach, since it gives the first two decimals in an exact same form: 3, 14. The class culminates with a verification activity that consists of: each group must designate a member to go to the blackboard carrying only a tape of equal length to the diameter of the disc of its group and he must cut an adherent paper tape exactly to the length of the perimeter of their disc. Then he returns to his group and verifies that it is the right measurement to cover exactly the contour of the disc, without lacking or going over adherent tape. The group that does it well, wins. The class was called: "How much does a wheel move in one turn? The students learned to quantify the length of a complete turn of a wheel. From the point of view of modeling (one of the key mathematical competences), we can say that in this real world problem, a mathematical structure was imposed on it: $P = 3\ 1/7 \cdot d = \pi \cdot d$, formula that relates the perimeter of the circumference with the length of the diameter.

- **Possible issues for discussion and reflections with teachers observing this lesson**

1. What may be the goals of this lesson?
2. How can we characterize the mathematics of this lesson?
3. How does the teacher view his students?
4. What are the characteristics of the classroom management of this teacher?
5. Is there more mathematics stakes in this problem of which the teacher should be aware?
6. What may be the learning outcomes and the follow-up for such lesson?

MATHEMATICAL THINKING IN MULTIPLICATION
IN HONG KONG SCHOOLS

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Introduction

The development of multiplication is a progress that allows students to learn a lot of patterns and also mathematics structures. This paper is based on a research of classroom teaching in mathematics of primary 3 to primary 5 students in Hong Kong. The same set of questions used in a primary 3 to primary 5 classes (except the question on Pissa). With the design of different sets of worksheets on non-routine problems, the development of concept of counting, repeated counting, multiple with counting, using multiple and then the law of multiplication to solve the problems is discussed. The results inform us that children using multiplication as a tool to solve questions of combinatoric nature (such as number of different grid formed by using different number of colour etc), are more difficult to understand than we thought. And the jumping of the cognitive gap from repeated counting and addition to multiplication needs certain daily examples to act as correspondence in concepts formation.

The using of the designed worksheets helps students to develop their concept of multiplication and also mathematical thinking through the connection of concepts by overcoming cognitive gaps.

The set of question used

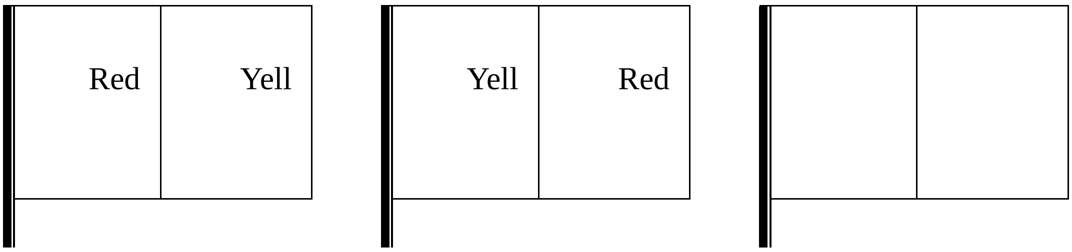
SET 1

Question A flag has 2-grid, if you have k colours to fill in the grids, and only each one of the colour is used once. How many different way are possible?

The worksheet has the following smaller questions. This is the format of all the sets of papers.

1	A flag has 2-grid, only each of the colours red and yellow can be used once. How many different way are possible?

2	A flag has 2-grid, only each of the colours red and yellow can be used once. How many different way are possible?
3	A flag has 2-grid, only each of the FOUR colours can be used once. How many different way are possible?



Students can find out that using FOUR colours can give them 12 different ways.

A	B	C	D	E
AB AC AD AE	BA BC BD BE	CA CB CD CE	DA DB DC DE	EA EB EC ED
5	5	5	5	5

By listing the table, some students are able to use the multiplication 5×4 = 20 to obtain the answer.
This is a systematic counting.
Can students jump from the results of 5 colours to 6 colours?
Many students still relay on the listing of the table for using SIX colours.

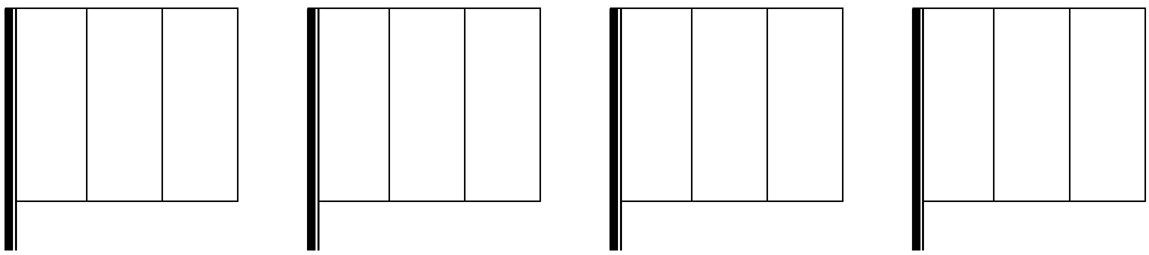
For primary 5 students, many can see that they have (k-1) choices for the first colour and k choices for the second colour. This is interesting that they think in the following process. If repeating the colours are allowed, for 7 colours, there will be 7×7 = 49 ways, but the answer is “(7-1)×7 = 42, for it is not allowed to repeat the colours”. They use (7-1)×7 = 42 rather than 7×(7-1) = 42. The same for the case of EIGHT colour, (8-1)×8 = 56 and not 8×(8-1) = 56.

Summary on 2-grids :	
Number of Colour	**Number of ways**
2	
3	
4	
5	
6	
7	
8	
k	

It is very difficult for students to generalize the case to “k” colours. Even for the primary 5 students, they are not able to see that the answer is (k-1)×k.
However, given a large value of k, say k = 100, some students can provide the answer of 100×99.

SET 2 using of 3 grids and k colours.

> Question
> A flag has 3-grid, if you have k colours to fill in the grids, and only each one of the colour is used once.
> How many different way are possible?



Since there are three grids, the listing is more difficult than the worksheet in SET 1. Students start to use A, B, C to represent colours after some hints and they use listing to count the answer. This is difficult to use table and ABC to count 4 colours and the following is how some students tackle the problem.

For FOUR colours (A, B, C, D), the first colour A can give 6 options

A	B	C	D
ABC ACB ABD ADB ACD ADC			
6	6	6	6

Thy use counting for the first colour (A), and then they know that it will be the same for the other colour. Hence they got the answer 6 + 6 + 6 + 6 = 24.

And a while later, students formulate the results for $6\times4 = 24$.

Similarly, they do the same for FIVE colours in 3-grids. But still students could not obtain the relationship of $5\times4\times3 = 60$.

However, many can fill in the following table up to case 8.

Summary on 3-grids：	
Number of Colour	Number of ways
3	
4	
5	
6	
7	
8	
k	

SET 3 Using number to fill in the grids

Question There are 3-grids. Insert the number 1, 2, 3, …, k into the grids once. How many different ways are possible?

The first question is using three numbers 1, 2, 3.

1	2	3

1	3	2

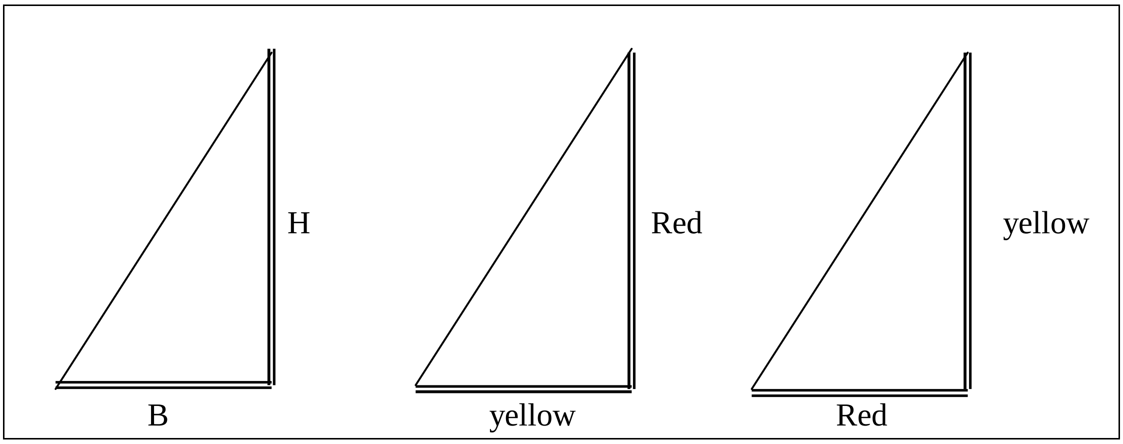
This set of papers is used in another class of primary 4. It is interesting that this problem, which does not have the context of a flag (grid), allow students to be more focus on the multiplication part.

The process of solution is similar. They list the cases and count the number. More students can obtain the results more quickly in compare to the answer in SET 2.

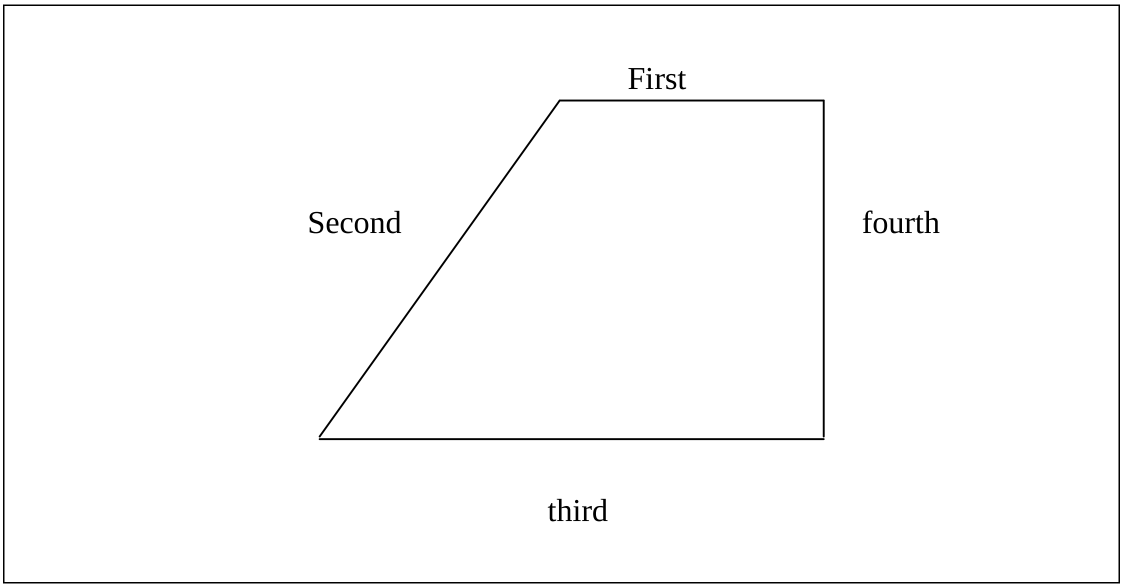
SET 4 Using a context of triangle

> Question
> If you can only use the colours “red”, “yellow” and “blue” for the triangle, each side uses only one colour, how many different ways are possible?

Students use table to obtain the answer quickly for triangle and quadrilateral.



Ways	Base colour	Height colour
1		
2		
3		
4		
5		
6		
7		
8		



Quadrilateral	
Number of colour	Different number
4	24
5	120
6	360
7	840
8	1680
9	3024
k	(k-1)(k-2)(k-3)k

However, for pentagon, no primary 5 students can obtain the general for of the result (k-1)(k-2)(k-3)(k-4)k.

Pentagon	
Number of colour	Different number
5	120
6	
7	
k	(k-1)(k-2)(k-3)(k-4)k

SET 5 Pissa toppings

Question You can choose a number of toppings for a pissa. How many different kind of pissa are possible if there are k different choices toppings?

ways	Toppings		
	Cheese	Beacon	Sausage

For the case of two to three toppings, students can obtain the answer easily. They do that by counting. Students can use counting to handle the question up to 4 choices of serving. For 5 or more toppings, students' counting can be lost and correct answer can only be obtained through the using of multiplication.

For 4 toppings, many students can get 16 ways. Their strategy is by using ticks in the table.

Toppings	1	2	3	4
	✓	✓	✓	✓
	✓	✓	✓	✗
	✓	✓	✗	✗

Both primary three to primary five students use counting to solve the problem. However, after using three toppings, the counting process is tedious. For example, students make mistake in using four toppings and give out answer such as 15 or 12. They have missed some cases in their counting.
Those who counted correctly try to use the pattern of the first few answer to obtain the solutions. Students do think that answer must satisfy the pattern they discovered, that is, in the sequence of 2, 4, 8, 16, 32 etc.

The first thinking process is a recurrence relation. If there are 4 ways for two toppings, and when one more topping is added, then there are two possibilities for the case of two toppings. The first one is no third topping is needed in this 4 cases, and the second one is the adding of the third toppings for the previous 4 ways. So students based on 4 + 4 = 8 to get their answer and not based on $2^3 = 8$.

For example, the following is the answer of two toppings, 4 ways.

	A	B
1	✓	✓
2	✓	×
3	×	✓
4	×	×

Then they repeated the above pattern, adding the third toppings, by giving four "✓" and four "×"., getting the answer 4 + 4 = 8.

	A	B	C
1A	✓	✓	✓
2A	✓	×	✓
3A	×	✓	✓
4A	×	×	✓
1B	✓	✓	×
2B	✓	×	×
3B	×	✓	×
4B	×	×	×

They need the answer of two toppings to obtain the answer of 4 toppings, and the answer of 5 toppings to get their answer of 6 toppings etc. They know that the "second" answer is a double of the first answer. However, they could not directly get the answer of 6 toppings from multiplication.

However, some students found that they can formulate question into either taking each of the toppings or not taking the toppings. There are two choices for each topping and their results are summarized in the following table.

Toppings	1	2	3	4	5	6
Yes	✓	✓	✓	✓	✓	✓
No	✗	✗	✗	✗	✗	✗
Number	2	2	2	2	2	2

So he found that the answer is $2\times2\times2\times2\times2\times2 = 64$.
The totally abstract use of multiplication is based on the understanding of the problem, familiar with the multiplication table and also the relation of repeated addition.

Summary

The above teachings show that it is difficult for students to formulate the solutions through multiplication. The thinking process of addition, repeated addition and multiplication need certain space for students to overcome their cognitive gap. Through such discussion of the listing of table, students can think of different ways in solving the problem. Though some of their answer through counting is not correct, they can verify them after some classroom discussion.

LESSON STUDY ON MATHEMATICAL THINKING:
Developing Mathematical Methods in Learning the Total Area of a Right Circular Cylinder and Sphere as well as the Volume of a Right Circular Cone of the Indonesian 8th Grade Students

Marsigit, Mathilda Susanti, Elly Arliani

Yogyakarta State University, Indonesia

In this study, the researchers strived to uncover the aspects of students attempts in developing mathematical methods in learning the total area of a right circular cylinder and sphere as well as the volume of a right circular cone of the 8th grade students of Junior High School. The results of the research describe students' attempts in inductive thinking, analogical thinking, deductive thinking, abstract thinking, thinking that simplifies, thinking that generalizes, thinking that specializes, thinking that symbolize, thinking that express with numbers, quantifies, and figures. Students' mathematical methods can be traced through the schema of teaching learning activities.

INTRODUCTION

The Decree of Sisdiknas No. 20 year 2003 insists that Indonesian Educational System should develop intelligence and skills of individuals, promote good conduct, patriotism, and social responsibility, should foster positive attitudes of self reliance and development. Improving the quality of teaching is one of the most important tasks in raising the standard of education in Indonesia. It was started in June 2006, based on the Ministerial Decree No 22, 23, 24 year 2006, Indonesian Government has implemented the new curriculum for primary and secondary education, called KTSP "School-Based Curriculum". This School-based curriculum combines two paradigms in which, one side stress on students competencies while on the other side concerns students' learning processes. The School-Based Secondary Junior mathematics curriculum outlines that the aims of teaching learning of mathematics are as follows:

1. to understand the concepts of mathematics, to explain the relationships among them and to apply them in solving the problems accurately and efficiently.
2. to develop thinking skills in learning patterns and characteristics of mathematics, to manipulate them in order to generalize, to prove and to explain ideas and

mathematics propositions.

3. to develop problem solving skills which cover understanding the problems, outlining mathmatical models, solving them and estimating the outcomes.
4. to communicate mathematics ideas using symbols, tables, diagrams and other media.
5. to develop appreciations of the use of mathematics in daily lifes, curiosity, consideration, and to encourage willingness and self-confidence.in learning mathematics.

This is why the aim of mathematics education from now on is still urgently to promote mathematical method and to take it into actions. Above all, these lead to suggest that it needs to conduct classroom-based research to investigate the necessary driving factors towards students' ability to develop mathematical method.

THEORETICAL FRAMEWORK

Katagiri, S. (2004) insists that the most important ability that children need to gain at present and in future, as society, science, and technology advance dramatically, are not the abilities to correctly and quickly execute predetermined tasks and commands, but rather the abilities to determine themselves to what they should do or what they should charge themselves with doing. Of course, the ability to correctly and quickly execute necessary mathematical problems is also necessary, but from now on, rather than adeptly to imitate the skilled methods or knowledge of others, the ability to come up with student's own ideas, no matter how small, and to execute student's own independence, preferable actions will be most important. Mathematical activities cannot just be pulled out of a hat; they need to be carefully chosen so that children form concepts, develop skills, learn facts and acquire strategies for investigating and solving problems.

Mathematical method

Mathematical thinking has its diversity of simple knowledge or skills. It is evidence that mathematical thinking serves an important purpose in providing the ability to solve problems on one's own as described above, and this is not limited to this specific problem. Therefore, the cultivation of a number of these types of mathematical thinking should be the aim of mathematics teaching. Katagiri, S. (2004) lays out the followings as mathematical thinking related to mathematical method:

> inductive thinking, analogical thinking, deductive thinking, integrative thinking (including expansive thinking), developmental thinking, abstract thinking (thinking that abstracts, concretizes, idealizes, and thinking that clarifies conditions), thinking that simplifies, thinking that generalizes, thinking that specializes, thinking that symbolize, thinking that express with numbers, quantifies, and figures.

Questions for Eliciting Mathematical Method

Teaching should focus on mathematical thinking including mathematical method. Questions related to mathematical thinking and method must be posed based on a perspective of what kinds of questions to ask. Katagiri, S. (2004) indicates that quaestion must be created so that problem solving process elicits mathematical thinking and method. He lists question analysis designed to cultivate mathematical thinking as follows:

a. Problem Formation and Comprehension

1) What is the same? What is shared? (Abstraction)
2) Clarify the meaning of the words and use them by oneself. (Abstraction)
3) What (conditions) are important? (Abstraction)
4) What types of situations are being considered? What types of situations are being proposed? (Idealization)
5) Use figures (numbers) for expression. (Diagramming, quantification)
6) Replace numbers with simpler numbers. (Simplification)
7) Simplify the conditions. (Simplification)
8) Give an example. (Concretization)

b. Establishing a Perspective

1) Is it possible to do this in the same way as something already known? (Analogy)
2) Will this turn out the same thing as something already known? (Analogy)

3) Consider special cases. (Specialization)

c. Executing Solutions

1) What kinds of rules seem to be involved? Try collecting data. (Induction)
2) Think based on what is known (what will be known). (Deduction)
3) What must be known before this can be said? (Deduction)
4) Consider a simple situation (using simple numbers or figures). (Simplification)
5) Hold the conditions constant. Consider the case with special conditions. (Specialization)
6) Can this be expressed as a figure? (Diagramming)
7) Can this be expressed with numbers? (Quantification)

d. Logical Organization

1) Why is this (always) correct? (Logical)
2) Can this be said more accurately? (Accuracy)

RESEARCH METHOD

The study was aimed at promoting students to develop mathematical method in learning the total area of a right circular cylinder and sphere and also the volume of a right circular cone. The approach used in the study was descriptive-qualitative of Lesson Study in two classes: the 8th grade of Junior High School, class A and the 8th grade of Junior High School, class B. The design of the research included: preparation (PLAN), implementation (DO), and reflection (SEE). The instrument used for collecting data consists of questionnaire, interview, observation of the lesson, and VTR of the Lesson. The research was began with two series of discussions between teachers and lectures and followed by observing and reflecting two lesson activities in the class, as the following description:

LESSON PLAN I

Day	**:**	**Thursday, May, 24, 2007,**
Date	**:**	**07.00-09.00**
Junior High School	:	SMP NEGERI DEPOK II, Yogyakarta, Indonesia
Grade/Sem/year	:	8/Sem II/2007
Teacher	:	Siwi Pujiastuti SPd
Class	**:**	**A**
Number of Students	:	40
Standard Competency	:	To understand the characteristics of cylinder, cone,

sphere and to determine their measures.

Basic Competencies : To identify the formula of the total area of right circular cylinder; to identify the formula of the area of sphere.

Teaching Scenario : 1. Apperception
2. Developing concepts
3. Reflection and presentation
4. Conclusion and closing

VIDEOTAPED LESSON I Part A :

Aim : To identify the formula of the total area of right circular cylinder

	Introduction: **Problem Formation and Comprehension** - Teacher let the students observe given model of right circular cylinder *(concretization and induction)* - Teacher let the students identify the components of the right circular cylinder *(method of abstraction)* - Teacher let the students define the concept of right circular cylinder *(method of abstraction)* - During the whole class teaching process, teacher encouraged students' abstraction *(method of abstraction)*

	Group Work and Discussion: **Establishing a Perspective** a. Students employed concrete model to search the total area of right circular cylinder *(employing concrete model to express the concept and induction)* b. Students broke-down the model of right circular cylinder into its components: two congruent circles and one oblong. *(employing concrete model to express the concept and induction)* c. Students learned that the height of right circular cylinder is equal to the width of its rectangle; and the circumference of the circle is equal to the length of rectangle. *(employing concrete model to express the concept and induction)*

<table>
<tr>
<td></td>
<td>Group Work and Discussion:
Executing Solutions
- Students learned that the lateral area of right circular cylinder is equal to the area of its rectangle. (analogy of concept and induction)
- Students learned that the total area of right circular cylinder is equal to the area of its rectangle plus the area of its two circles. (analogy of concept and induction)</td>
</tr>
<tr>
<td></td>
<td>Students' Reflection :
- Students presented that the lateral area of right circular cylinder is equal to the area of its rectangle i.e.
LATERAL AREA = HEIGHT X CIRCUMFERENCE OF CIRCLE = $t \times 2\pi r$
- Some students needed to have clarification that the lateral area of right circular cylinder is equal to the area of its rectangle. (logical organization, analogy of concept and induction)
- Students presented that the total area of right circular cylinder is equal to the area of its rectangle plus the area of its two circles i.e.
TOTAL AREA = LATERAL AREA + 2 (AREA OF CIRCLE) = $t \times 2\pi r + 2(\pi r^2) = 2\pi r (t + r)$</td>
</tr>
</table>

VIDEOTAPED LESSON I, Part B:

Aim : to identify the formula of the area of sphere.

<table>
<tr>
<td></td>
<td>Introduction:
Problem Formation and Comprehension
- Teacher let the students observe given model of Sphere (Concretization and Induction)
- Teacher let the students identify the components of Sphere (Abstraction)
- Teacher let the students define the concept of Sphere (method of abstraction)
- Teacher's explained the way to find the area of the surface of Sphere.</td>
</tr>
</table>

Group Work and Discussion:

Establishing a Perspective

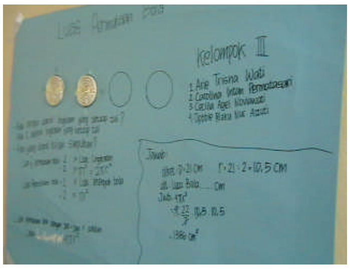
d. Students employed concrete model of a half Sphere to search the area of its surface *(employing concrete model to express the concept and induction)*

e. Students prepared the supporting facilities e.g. diagram of circles that has similar radius with the Sphere. (employing concrete model to express the concept and induction)

Group Work and Discussion:

Executing Solutions

- Students learned that the area of Sphere is equal to the area of its cover *(logical organization, analogy of concept and induction)*
- Students learned that the area of the surface of a half Sphere is equal to the double of the area of its circles. *(logical organization, analogy of concept and induction)*



Students' Reflection :

- Students presented that the area of the surface of a Sphere is equal to four times the area of its circles.
  AREA of SPHERE = 4 X AREA of CIRCLE
- Some students needed to have clarification whether their formula was correct? *(logical organization, analogy of concept and induction)*

LESSON PLAN II

Day	**:**	**Saturday, May, 26, 2007**
Date	**:**	**07.00-09.00**
Junior High School	:	SMP NEGERI DEPOK II, Yogyakarta, Indonesia
Grade/Sem/year	:	8/Sem II/2007
Teacher	:	Siwi Pujiastuti SPd
Class	**:**	**B**

Number of Students : 40

Standard Competency : To understand the characteristics of cylinder, cone and sphere and to determine their measures.

Basic Competency : To identify the formula of the volume of right circular cone.

Teaching Scenario :
1. Apperception
2. Developing concepts
3. Reflection and presentation
4. Conclusion and closing

VIDEOTAPED LESSON II

Aim : To identify the formula of the volume of right circular cone.

Introduction:

Problem Formation and Comprehension

- Teacher let the students observe given model of right circular cone. *(concretization and induction)*
- Teacher let the students identify the components of the right circular cone. *(method abstraction)*
- Teacher let the students define the concept of right circular cone. *(method abstraction)*
- During the whole class teaching process, teacher encouraged students' abstraction *(method of abstraction)*

Group Work and Discussion:

Establishing a Perspective

f. Students learned the teacher's guide to understand the procedures how to search the volume of right circular cone. *(employing concrete model to express the concept and induction)*

g. Students observed and manipulated the concrete model of right circular cone and right circular cylinder *(employing concrete model to express the concept and induction)*

	Group Work and Discussion: **Executing Solutions** - Students collected data of the measurement of the volume of cone in comparison with the volume of cylinder. *(logical organization, analogy of concept and induction)* - Students learned that the volume of right circular cylinder is equal to three times the volume of right circular cone. *(logical organization, analogy of concept and induction)*
	**Students' Reflection :** - Students presented that the volume of right circular cone is equal to one third of the volume of right circular cylinder. *(logical organization, analogy of concept and induction)* - Some students needed to have clarification whether their formula was correct *(logical organization, analogy of concept and induction)*

ANALYSIS OF DATA

a. Problem Formation and Comprehension

The students manipulated *Concrete Mode*l of the *Right Circular Cylinder, Sphere and Right Circular Cone* in order to identify its components. They performed *mathematical abstractions* when the teacher gave them some questions or when the teacher let them work in group. Some students defined the *concept* of *Right Circular Cylinder* as its *functions* in daily life e.g. *"A Right Circular Cylinder is the storage to keep something like pen, pencil, etc."* The teacher encouraged the students to perform *mathematical abstractions* of the *Concrete Mode*l of the *Right Circular Cylinder* viz. to indicate its components such as circles, height, and the radius of its circle. There were students who defined *a Sphere* by giving the example of daily life e.g. ball, tennis-ball, etc. Students' *abstractions* of Sphere resulted the investigation of its components i.e. the *radius* and *diameter*.

At the first step, most of the students defined the right circular cone through *characterization of its shape,* e.g. *"A right circular cone is a thing like triangle", "A right circular cone is composed by three dimensional triangle",* and *"A right circular cone is composed by many circles – the higher the circle the smaller it does".* There were many ways in which the students *idealized* the geometrical concept. They mostly confirmed the concept to the teacher and asked their mates. Sometimes they performed their *idealization* by commenting other works. Some students asked the teacher why the *lateral area of cylinder is equal to the area of its rectangle* and why the *volume of cylinder is equal to three times the volume of its cone.*

b. Establishing a Perspective

Working in group triggered the students to develop analogical thinking of mathematical concepts. *Analogical thinking* happened when the students perceived that finding the lateral area of *Right Circular Cylinder* is similar to finding the area of its rectangle; and, finding the area of *Sphere* is similar to finding the area of its surface i.e. covering its surface by twisting around the rope. In sum, the concepts of geometrical shapes are mostly perceived to be analogical with examples in daily life e.g. the *right circular cone was perceived* as a *traditional hat.* In performing their *analogical thinking* the students frequently used strategic terminologies such as *"similar to", "comparison with", "the example of",* and *"the function of".*

Most of the students perceived that the given task by the teacher as the cases that need special consideration. Therefore, most of them considered more seriously on the ways to find the formulas of the total area of right circular cylinder and the formula of the area of sphere as well as the formula of the volume of right circular cone. Some students still paid attention on the concepts of cylinder, sphere, and cone. There was a student who wanted a clarification on the form of circle bases of cone whether it is convex, concave or plane. After getting input from teacher or their mates, the students usually *considered special cases* including corrected the formulas and made some notes at their works. Some students *considered* the use of $\pi = \frac{22}{7}$ *or* $\pi = 3.14$. One student expressed that for the bigger radius, we use $\pi = \frac{22}{7}$ and for the smaller radius we use $\pi = 3.14$.

c. Executing Solutions

Students' *inductive thinking* involved *Concretization and method of abstraction* in the area of *Problem Formation and Comprehension.* When the students who known the certain concept, were paced to perform *inductive thinking* they tend to reconfirm their concepts. *Inductive thinking* was spread from the beginning activities to the ultimate accomplishment when the students were paced to do so. The students observed the given model of right circular cylinder and strived to identify the components of the right circular cylinder in order to define the concept of right circular cylinder (method of abstraction). Students' *inductive thinking* was also related to *establishing perspective* in which the students employed concrete model to search the total area of right circular cylinder and broke-down the model of right circular cylinder into its components: two congruent circles and one oblong.

There were some steps in which the students perform *inductive thinking*, as following:

Inductive thinking* of finding the total area of *Right Circular Cylinder:

Step 1: Observing the *Concrete Mode*l of *Right Circular Cylinder*
Step 2: Manipulating the model and learning the teacher's guide
Step 3: Drawing the components of the *Right Circular Cylinder* i.e. the bottom circle, the top circle and its rectangle.
Step 4: Determining the area of its components
Step 5: Adding up the total area

Inductive thinking* of finding the area of *Sphere:

Step 1: Observing *Concrete Model of a half Sphere*
Step 2: Manipulating the model and learning the teacher's guide

Step 3: Twisting around of the model with the rope
Step 4: Thinking inductively that the length of rope needed to twist around a half model of sphere determines the area of circle.
Step 5: Finding out that the area of a half sphere is equal to two times the area of its circle.

Inductive thinking* of finding the volume of *cone:

Step 1: Observing *Concrete Model of Right Circular Cone and its Right Circular Cylinder*
Step 2: Manipulating the model and learning the teacher's guide
Step 3: Practicing to fill up the *Right Circular Cone* with *sand* which was acted of measuring out by the *Right Circular Cone*
Step 4: Thinking inductively to find out the volume of *Right Circular Cone* compare with the volume of its *Right Circular Cone*
Step 5: Finding out that the volume of the *Right Circular Cone* is equal to one third of the volume of its *Right Circular Cylinder*

d. Logical Organization

Logical organization of mathematical concept happened in all context of mathematical method: *idealization, abstraction, deduction, induction and simplification. Logical organizations* of mathematical concept can be indicated from the following example of students' questions:

1. Why is the lateral area of cylinder equal to the area of its rectangle?
2. Why is the volume of cylinder equal to three times the volume of its cone?
3. What happens if we do not carefully cover the surface of the sphere in which we use the rope for twisting around?
4. Is it true that the area of the surface of sphere is equal to 4 times the area of its circle?

DISCUSSION

Krygowska (1980) in Bonomo M.F.C indicates that mathematics would have to be applied to natural situations, any where real problems appear, and to solve them, it is necessary to use the mathematical method. The knowledge, skills, and mathematical methods are the foundation to achieve the knowledge on science, information, and other learning areas in which mathematical concepts are central; and to apply mathematics in the real-life situations. This study uncovered that teacher has important role to encourage their students to develop mathematical methods. The students performed *mathematical method* when they found difficulties or when they were asked by the teacher. Most of the students reflected that they paid attention on the perfect of the *Concrete Model* of geometrical shape. However, their consideration on the perfect form of the models did not indicate that they performed *mathematical idealization* as one of mathematical method.

Student's reflection:

Researcher	:	What is the effect on your calculation of the volume if a *right circular cone* is made up from *"a very thick metal plat"*
Student	:	It's okey. No, problem.
Researcher	:	Compare to a *right circular cone* if it made up from *"a very thin metal plat"* How can you determine the volume?
Student	:	In "*"a very thin metal plat cone"* I can fill up *more sand.* But if the radius and the height of the cone are similar, they should have similar volume.
Researcher	:	So, what do you think about the different between the cone that made up from *"a very thick metal plat"* and cone that made up from *"a very thin metal plat"* ?
Students	:	Yes, it will be very different. I am sorry for my initial statement.

Meanwhile, David Tall (2006) states that success in mathematical thinking depends on the effect of met-befores, the compression to rich thinkable concepts, and the building of successive levels of sophistication both powerful and simple. In this research, one aspect of mathematical method i.e. s*implifications* happened when the students perceived that the concept of *right circular cone* is similar to the concept of triangle or circle. In this case, they *simplified* the concepts through manipulation of *Concrete Models.* They also performed simplification when they broke down the formula to solve the problems. They mostly simplified the concepts when they had got some questions from the teacher; or, when they worked in group. Ultimately, when the teacher asked for the students to write the results, the students got that: 1) the total area of *Right Circular*

Cylinder is equal to the area of its rectangle plus two times the area of its circles, 2) the area of *Sphere* is equal to four times the area of its circle, and 3) the volume of *right circular cone* is equal to one third of the volume of its cylinder.

The students developed *inductive thinking* when they uncovered that the height of right circular cylinder is equal to the width of its rectangle; and the circumference of the circle is equal to the length of rectangle. They continued to perform *inductive thinking* until they found the formula of the lateral area of right circular cylinder; the formula of sphere, and the formula of the volume of *Right Circular Cone.* Students' *schema of inductive thinking* seemed in line with Katagiri's claim that *inductive thinking* covers:

> 1) Attempting to gather a certain amount of data, 2) Working to discover rules or properties in common between these data, 3) Inferring that the set that includes that data (the entire domain of variables) is comprised of the discovered rules and properties, and 4) Confirming the correctness of the inferred generality with new data

In a different context, Stacey, K (2006) suggests that a key component of mathematical thinking is having a disposition to looking at the world in a mathematical way, and an attitude of seeking a logical explanation. While Katagiri, S. (2004) claims that students' logical actions include: attempting to take actions that match with the objectives; attempting to establish a perspective; and attempting to think based on the data that can be used, previously learned items, and assumptions. In this research, the aspects of *logical organization* of mathematical concept emerged after the students put into practice mathematical procedures in their group. However, there were evidences that it was difficult for the students to practice mathematical procedures. Students' inappropriate organization of mathematical procedures appeared when the students had difficulties in performing mathematical procedure into practice with *concrete mode*l. In searching the formula of the volume of right circular cone, there were some students who hesitated what to fill-up with sand. Should it be a cone or cylinder? In searching the formula of the total area of *a right circular cylinder,* there was a question from the student, why the total area is the result of addition and not the result of multiplication.

CONCLUSION

In this Lesson Study, the researchers had sought to uncover the picture in which the teacher strived to promote mathematical methods in learning the total area of a right circular cylinder and sphere as well as the volume of a right circular cone. The striking results of the study can be stated that *students' mathematical methods* can be traced through the schema of teaching learning activities as follows:

1. Problem Formation and Comprehension were emerged when the students:
   a. observed given model of right circular cylinder, observed given model of Sphere, and observed given model of right circular cone
   b. identified the components of the right circular cylinder, sphere, and right circular cone
   c. defined the concept of right circular cylinder, sphere, and right circular cone
   d. got questions and notices from teacher to search the concepts
2. Establishing a Perspective were emerged when the students:
   a. employed concrete model to search the total area of right circular cylinder, the area of sphere and the volume of right circular cone
   b. learned that the height of right circular cylinder is equal to the width of its rectangle; and the circumference of the circle is equal to the length of rectangle
   c. learned the teacher's guide to understand the procedures how to search the volume of right circular cone
   d. broke-down the model of right circular cylinder into its components
3. Executing Solutions were emerged when the students:
   a. tried to find out the lateral area of right circular cylinder
   b. tried to find out the total area of right circular cylinder
   c. tried to find out the area of sphere
   d. collected the data of the measurement of the volume of cone in comparison with the volume of cylinder

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MATHEMATICAL THINKING AND THE ACQUISITION OF FUNDAMENTALS AND BASICS

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Hokkaido University of Education, Japan

1. Fundamentals and Basics in Mathematics Education

The initial report of the Central Education Council issued in 1996 proposed that the formation of the "power for living," whereby one learns and thinks for oneself, in an educational environment free of pressure is the basis of education in the 21st century. Subsequently, based on a number of additional reports, the final report of the Central Education Council was issued, and the curriculum created accordingly along school curriculum guidelines went into effect in 2002. Under this curriculum, educators are required to help pupils form the capacity to identify issues on their own and proactively solve problems amid a society undergoing rapid change. According to school curriculum guidelines, fundamentals and basics are the basis that supports the various living and learning activities of children. For example, they are the basis of daily living activities, various school activities, the continuous learning of mathematics, and future social and lifelong activities. The guidelines emphasize the systematic acquisition of fundamentals and basics, through repeated study if needed, in order to enable the smooth pursuit of such activities.

The above approach to fundamentals and basics suggests a large number of issues related to current education and study guidance in the field of elementary school mathematics, such as the following:

A. An approach that combines guidance that emphasizes fundamentals and basics and nurtures individuality by enabling children to learn and think for themselves
B. Fundamentals and basics should be understood simply as formal guidance in terms of knowledge and skills, but they should include the aim of guidance in terms of the ability to think, judge, express oneself, and so on.
C. Activities to acquire fundamentals and basics should be understood in terms of children achieving goals through the autonomous study of problem solving, and fundamentals and basics should be considered to include study methods and the ability to solve problems.

The formation of the ability to learn and think independently should be understood in terms of the fundamentals and basics that make up the core that develops the drive to learn independently and the ability to proactively adapt to changes in society, in other words, the "power for living." Naturally, guidance for such a learning approach should be thought of in terms of not only textbooks but also the entire school spectrum and activities. Here, we examine the ability to learn independently, as fostered through the study of elementary school mathematics in particular.

The development of children who learn independently requires, first of all, teachers who are sufficiently aware of the importance of such endeavor and rigorous consideration of the type of guidance required to this end.

Furthermore, the ideal form of mathematics classes to teach fundamentals and basics to pupils has been considered based on the above. Fundamentals and basics are not just achieved through a one-hour-a-week class but need to be acquired over many hours (throughout the school year or throughout the course unit). The study of elementary school mathematics is characterized by a spiral-shaped progression of various component areas, with each area having a distinct study style, and it is necessary to promote practical research in the aspects of fundamentals and basics as well as the ideal guidance approach for each area, the problem-solving process per unit time, and the interrelation and realization of learning fundamentals and basics. In this paper, the author considers an approach to mathematics particularly from the aspect of fundamentals and basics.

2. Approach to Mathematics from the Aspect of Fundamentals and Basics

The aims of elementary school mathematics in Japan are the acquisition of fundamental and basic knowledge and skills regarding quantities and geometric figures and, based on this, the cultivation of the basics of creativity—such as the ability to think about things from a number of different aspects and the ability to think logically—as well as the development of an understanding of the merits of investigating and handling phenomena mathematically and the attitude of applying the knowledge thus obtained to subsequent objects of study or daily life. These aims include the definition of the type of mathematics instruction to be realized in the classroom. In elementary school mathematics, the cultivation of the ability to solve problems has been promoted for the formation of "how to learn." For the type of problem solving aimed for in elementary school mathematics, what is expected is the formation of the attitude of developing, through the solving of problems, new ways of looking and thinking about things that can be shared with others in the class and, in this manner, improve one another and achieve self-growth. The formation of how to learn demanded by the Ministry of Education, Culture, Sports, Science and Technology has been achieved based on the school research theme of developing the ability to solve problems. In this sense, in Japan's elementary school mathematics, developing the ability to learn and developing the ability to solve problems are considered to be almost synonymous.

Normally, when learning problem-solving skills, pupils progress through five stages. From the viewpoint of the formation of how to learn, the fundamentals and basics to be taught at each stage are as follows:

(1) Problem setting
Stage at which problems for study are created through the introduction of course units, class periods, etc.
- Ability to set learning targets and set learning sequences
- Ability to grasp the aims of the teacher's lesson progression
- Acquisition of the mathematical way of thinking

(2) Problem solving
Stage at which pupils grasp the situation and understand what the problem is
- Ability to form one's own question from the problem (visualization of situation and formulation of questions)
- Ability to share information about the problem with classmates
- Ability to engage in analogical reasoning from previous learning

(3) Solution planning
Stage at which pupils determine the direction required for solving a problem that has been understood
- Ability to recall previous learning contents and experience (thinking in terms of units, etc.)
- Ability to make predictions (prediction of meaning of calculation and estimation of solution)
- Ability to attempt problem solving on one's own (reexamination of calculation method)
- Intuition ability

(4) Solution execution
Stage at which a solution to a problem is attempted based on the solution plan and a tentative conclusion is drawn
- Ability to execute a solution based on a plan
- Ability to utilize past learning experience and contents (deduction from past learning)
- Ability to express one's thoughts in a manner understandable to classmates (actions

of thought,
judgment, and expression)

- Ability to recognize differences between one's own and one's classmates' thinking and to clarify the essence forming the background of these differences
- Ability to look back on one's actions (functional solution)

(5) Solution study (including review)
Stage at which pupils adequately evaluate processes and achievements and clarify what they have understood and what should be further investigated

- Ability to sympathize with the views of others, going beyond differences in thinking
- Ability to study solutions and come up with better results (perception of integration and development, refining of solution, integration with one's past learning, and acquisition of knowledge and skills)
- Ability to make generalizations and developments (generalization of a line of thought, abstraction, and logical processing)
- Ability to compare what has been learned in class with one's own and one's classmates' thinking and to evaluate oneself (validity of estimation)

The fundamentals and basics at these stages involve many different aspects, such as interest-, motivation-, and attitude-related aspects; mathematical thinking-related aspects; thinking ability-; judgment ability-; and expression power-related aspects; and knowledge- and skill-related aspects.

This section describes the mathematical way of thinking. The mathematical way of thinking is considered to form the core of mathematical education and the base from which arithmetic and mathematics knowledge and skills are produced. Let us examine the approaches discussed below, focusing on school year development equivalents in relation to children's awareness of the benefits and their application thereof through the learning contents in each area.

(1) Mathematical approach related to content—concept formulation, principles and rules
(2) Mathematical approach related to method—thinking in a mathematical manner
(3) Mathematical approach as a logical way of thinking—thinking in a logical manner
(4) Mathematical approach as an integrated and developmental way of thinking—thinking in an
integrated and developmental manner
(5) Mathematical approach that promotes a mathematical sense and thinking ability—numerical sense, quantitative sense and graphic sense
(6) Mathematical approach that promotes a mathematical attitude—approach to problem solving

- Concept formation, principles and rules: Thinking in terms of units, thinking in terms of place value numeration, thinking in terms of correspondence, percentages, etc.
- Thinking mathematically: Idealization, encoding, simplification, formalization, compaction, etc.
- Thinking logically: Analogic thought, inductive thought, deductive thought, etc.
- Thinking in an integrated and developmental manner: Abstraction, generalization, expansion, etc.
- Sensory and thinking ability: Estimation, sense of volume, approximation, etc.
- Arithmetic and mathematical attitude: Utilization of previous learning, outlook, perception of value, etc.

Interest, motivation, and attitude stimulate the intellectual curiosity of the child and, as such, are important motivating forces in the study of mathematics. Each one of these is considered a mental disposition that is actively expressed on one's own from the viewpoints of "the mathematical way of thinking and expression, processing, knowledge and understanding" and

“ways of learning and problem-solving skills.” These are positioned as “items that support the fundamentals and basics of content and method.” The learning process of pursuing the value and significance of mathematical value is necessary in developing mental disposition.
In the following section, we will focus more particularly on fundamentals and basics related to introduce practices.

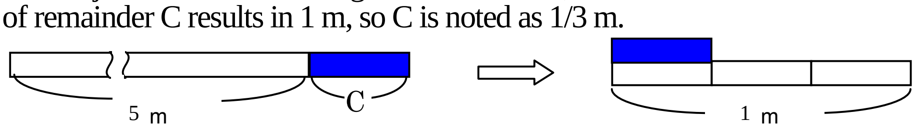
3. Formation of the mathematical way of thinking related to content: The case of thinking in terms of units

With regard to the formation of the mathematical way of thinking discussed in this paper, let us examine the way of thinking about units, from the aspect of content in particular. Let us consider the concept of units when studying the concept of numbers in the area of numbers and arithmetic (integers, decimals and fractions) and units during the numerical conversion of amounts related thereto.

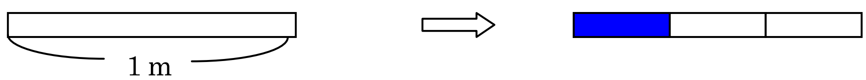
1) **Formation of the concept of units during the number concept formation process**

The acquisition of the number concept begins in the first year, and by looking at things from the same viewpoint, through activities in which one group is created and through one-on-one correspondence between two or more groups, pupils learn numeric representation and relations of magnitude among numbers through division into classes among these groups and through the number of class factors. Moreover, in learning about amounts and estimation—which are closely related, as will be noted later—first-year pupils form the foundation of later numerical linear algebra, as expressed by “length equivalent to x number of erasers,” to take length as an example. Here, one piece is used as the unit. In their second year, pupils learn multiplication. In the case of “4 times 3,” 3 is seen as a unit of addition repeated 4 times. In this case, the fact that numbers 1 through 9 are each units that can be counted is used. Such activities are the basis of pupils’ understanding of the concepts of decimals and fractions over four years of instruction.
Decimals are introduced in the fourth year as numbers used in expressing the fractional part in relation to the estimation of amounts. For example, when performing a measurement using a 1-m ruler, when the measured length is “1 m plus,” based on the concept of decimal notation, that “plus” fraction is processed with the concept of dividing the unit (1 m) into 10 equal parts and then using one of these 10 subdivisions as the new unit, the pupil learns numerical conversion of fractions into a number of such new units based on this.
Moreover, when measuring amounts using unit quantities—unlike in the case of decimals, where the decimal notation system is the basis for expressing fractional amounts—the introduction of new numbers and fractions comes to mind. The following two methods come to mind as such a method. Let us consider, for example, the case of 1/3.

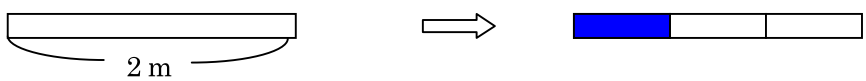
□ When an object is measured using 1 m as the unit, remainder C remains. Combining three of remainder C results in 1 m, so C is noted as 1/3 m.



□ Let us divide 1 m by 3 to obtain 1/3 m.



□ and □ are the same amount but are different ways of thinking. This difference is clearly shown in □’ below.
□’ 2 m is divided by 3, and 1 segment is denoted as 1/3 m (incorrect).



Both □ and □ can be called fractions that express fractional quantities in the sense that they

express an amount, but □ is an amount representation in the extension of fractions of the part-whole by equal splitting (several equivalent parts obtained by splitting the whole into equal parts) and leads easily to such a concept as that in □'. In the case of fractions of the part-whole by equal splitting, the whole is regarded as 1, and the action of splitting this whole into equal parts is emphasized, whereas when obtaining the number of split parts by dividing the unit amount of 1 m by the remainder amount and expressing the size of one such part, the remainder amount becomes the unit. In approach □, the approach of a given number of the unit fraction becomes clear. The approach of expressing 1 m with the remainder in □ is demonstrated by the Euclidean algorithm. Actually, this approach lies in obtaining the greatest common divisor c in relation to two natural numbers a and b[1].
The arithmetic action of using the remainder that remains following division as a unit that is again used for measurements is clearly different from the action of splitting the unit in □ into equal parts and using the expression as several of these parts. Moreover, it is important to consider such decimals and fractions as 0.2 and 2/7 in terms of two new units of 0.1 and 1/7, respectively. In this manner, the approach of thinking in terms of a new unit and creating such a unit plays a very important role when thinking about the four arithmetical operations for decimals and fractions. In other words, let us consider the sum of decimals and fractions as follows.

Content of "Amounts and Measurements" That Are Being Taught

In Japan, pupils learn the numerical conversion of amounts using units when being taught "amounts and measurements," broken down by school year, as follows. Such a structure aims to have pupils learn about the existence of various amounts and corresponding units as well as develop an awareness of units.

First year: Lengths (direct comparison, indirect comparison and measurements using arbitrary units)

Second year: Lengths (meaning of units and measurements and measurements using universal units (cm, mm and m))

Third year: Length (ability to make estimates and measurements using universal units (km))

Bulk and volume (concepts and measurements using universal units (l, dl and ml))

Time of day and time (duration) (concept, units (day, hour, minute and second), obtaining the time of day and time (duration))

Fourth year: Extent and area (concept, area of squares and rectangles, measurements using universal units (cm^2, m^2 and km^2) Angle size (concept and measurements using a universal unit (degree (°))

Fifth year: Quadrature formula for areas (areas of a triangle, parallelogram and circle)

Sixth year: Bulk and volume (concepts of volume, cubes, rectangular parallelepiped and measurements using universal units (cm^3, m^3)

Regarding the various types of amounts, in the measurement stage, pupils learn numeric conversion into a number of that given unit (arbitrary unit or universal unit). However, to make pupils understand the meaning of such measurements, it is important to make them practice repeatedly, using arbitrary units. Through the repeated implementation of such instruction, pupils deepen their understanding of units.

1. The Euclidean algorithm is a method of certifying the mathematical world by performing measurements with units. Assuming natural numbers a and b, with a > b, we obtain $a \div b = q_1 + r_1$ ($0 □ r_1 < b$). If $r_1=0$, we obtain the greatest common denominator b. If $r_1 \neq 0$, $b \div r_1 = q_2 + r_2$ ($0 □ r_2 < r_1$), and if $r_2=0$, we obtain the greatest common denominator of a, b and r_1. If $r_2 \neq 0$, the same operation is repeated, so that when $r_1 \div r_2 = q_3 + r_3$ ($0 □ r_3 < r_2$) , …… , $r_n \div r_{n+1} = q_{n+2}$ results, the greatest common denominator of a and b is r_{n+1}.

2) Case: Formation of the mathematical way of thinking in courses introducing decimals

The introduction of decimals is discussed using cases (see appendix). The instructors are trainee teachers (third-year students), yet they can be called good instructors even when compared to currently active teachers. The inculcation of many different ways of thinking is sought in one class. In this class, the instructor aims to form the following types of mathematical thinking, including the concept of units.

□ Understanding the concept of decimals (Concept formation: Concept of decimals)

By the time they start the course, pupils have learned the concept of integers greater than 0, the meaning of the four arithmetic operations and how to perform calculations. During the course, through the use of fluid volumes, pupils learn decimal notation using new units for amounts that cannot be expressed with integers using l as a unit. The point at which ingenuity is used in the course is when instructors have pupils perform the numerical conversion of amounts while keeping them interested by preparing the same amounts of liquid in different containers by group and using water of different colors. The pupils are made to write the amounts using cups listed in the worksheets provided by the instructor and are taught in an easy manner through activities in which they perform numerical conversion of odd amounts.

Using the fact that 1 dl, a unit the pupils have learned by then, is one tenth of 1 l, pupils are shown that this tenth is expressed as the decimal 0.1 in relation to the original 1 l unit, and they learn that 2 dl can be represented as 0.2 l.

□ Forming an attitude that facilitates making estimates (Sense and thinking ability: Estimates)

One more important thing about this course is that, after providing cups containing colored water, the teacher should ask the pupils, “How much water do you think they contain?” When learning about arithmetic and mathematics, it is important to acquire a sense of quantity.

□ Expressing amounts with 1 unit (Thinking mathematically: Simplification and integration)

From what they have learned up to this point, pupils have acquired the understanding that the amount obtained is “1 l and 2 dl,” but they must learn how to express this amount more simply using just one unit: l. In learning arithmetic and mathematics, pupils must understand that the simplest expression is desirable.

4. Mathematical Tools That Must Be Provided to Children

The term *mathematics class* may be misconstrued as practicing calculations. In a class where one learns and thinks for oneself, the mathematical way of thinking with regard to method and the mathematical way of thinking with regard to content are simultaneously acquired by the child. The implementation of problem-solving classes consists in allocating various problem-solving methods throughout the entire class, making pupils learn and accept each other's way of thinking, thereby teaching them the benefits of thinking. During this process, children learn how to think in relation to content and method from having to find ways to express their thoughts to each other.

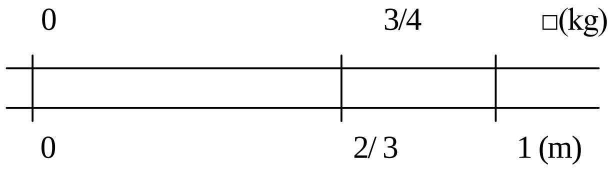
In order for children to become able to skillfully express content and method, they need to acquire the tools they will need when thinking about ways to express these things. Actually, as thoughts are exchanged, even when, for example, one feels that a tool (method of expression) or way of thinking used by a child in class is effective, there will likely be few opportunities to re-present situations so that all the children will be able to use such tools.

In the process of learning, the presentation of opportunities for the child to choose his/her own useful tools when thinking mathematically is important for forming the mathematical way of thinking. In so doing, it is also important to enable the child to use the same tools continuously and to develop these tools to create new ones. The term *problem solving* involves an emphasis on opportunities to make discoveries, but such discoveries, rather than being sudden occurrences, are for the most part achieved based on previous learning. If anything, it is important to value learning opportunities regarding ways to use tools as a part of the thinking process and ensure that children have an ample supply of such needed tools as they tackle problem solving.

For example, in the area of numbers in arithmetic, children can use line charts or surface diagrams to show relationships between quantities and explain what they are thinking. Even though they know that these line charts and surface diagrams are effective, there are probably many children who do not understand how to use these tools. In this sense, it is important to teach them how to use line charts and surface diagrams as tools for thinking. In the following, let us take up the case of two number lines as an extension of line charts.
Example: Let us consider the following problem.

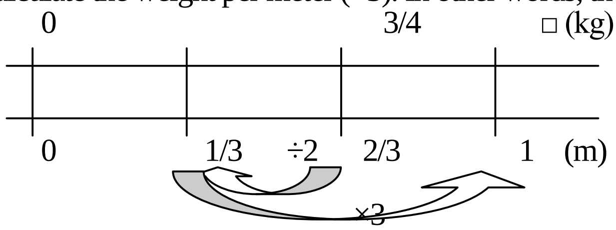
If an iron rod measures 2/3 m and weighs 3/4 kg, how many kg would it weigh if it were 1 m?

The following diagram can be used to understand that the formula for solving this problem is 3/4 ÷ 2/3 and that this is the means of solving this problem. (Because we used division to obtain the amount per meter in the integer problem, we shall use division here too.)

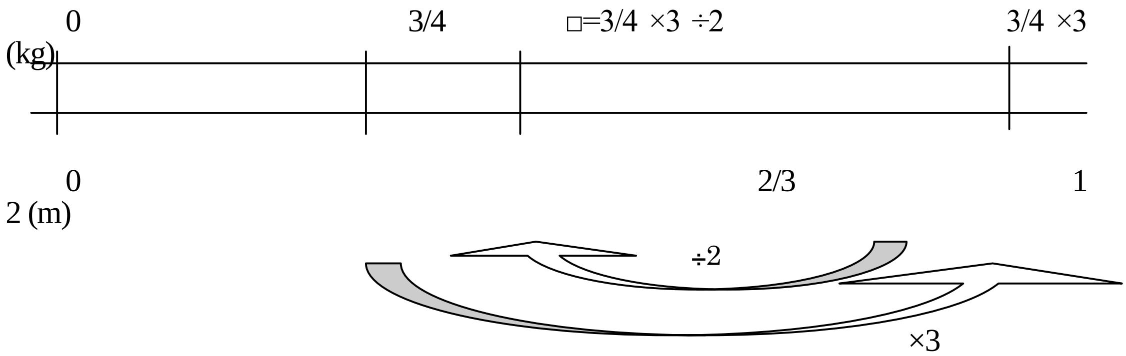


Using this diagram, let us think in the following way, for example, to obtain the weight of a 1-m iron rod.

 To obtain the weight per meter, first let us obtain the weight of 1/3 m (÷2) and then calculate the weight per meter (×3). In other words, the formula is 3/4 ÷ 2/3 = 3/4 ÷ 2 × 3.



 To obtain the weight per meter, first let us obtain the weight of 2 m (×3) and then calculate the weight per meter (÷2). In other words, 3/4 ÷ 2/3 = 3/4 × 3 ÷ 2.



In this manner, in classes that value children solving problems on their own, it is necessary that pupils learn how to use the tools that enable them to think for themselves. The ability to use the two number lines is acquired, by its very nature, through many hours of practice, and children that do not know how to use it at first gradually learn. Also, the two number lines require five or six years to master and are not something in which one can excel overnight. Textbooks are devised so that the line charts to be used by first and second year pupils are systematically and continuously addressed and are learned as an extension of line diagrams.

References:

Shigeo Katagiri, New Edition, “Mathematical Thinking and How to Teach It (I),” *Mathematical Thinking and How to Teach It*, Meiji Tosho (2004)

Shigeo Katagiri, “Mathematical Thinking and How to Teach It,” *Progress report of the APEC Project: “Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (□),”* University of Tsukuba (2007) pp. 105–158.

Kazuyoshi Okubo, “Mathematical Thinking from the Perspectives of Problem Solving and Area Learning Contents,” *Progress report of the APEC Project: “Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (□),”* University of Tsukuba (2007) pp. 237–244.

Appendix

Fourth Year Elementary School Mathematics Teaching Plan

Date & Time: September 18 (Thu.), 2003; 5th period
Pupils: Fourth-Year Class 3; 15 boys, 15 girls; total: 30
Instructor: Meien Elementary School , Sapporo City(Education Trainee) Fumito Chiba

1. Unit Name: Decimals
2. Unit Aims

To teach pupils the use of decimals to express the size of fractional parts that do not reach the unit quantity and have them use decimals appropriately

- Have pupils realize that the size of fractional parts that do not reach the unit quantity is expressed with decimals and have them readily try to use decimals.
- Have pupils realize that decimals, like integers, are expressed using the decimal notation system.
- Have pupils realize that, based on the decimal system, addition and subtraction operations can be done in the same manner as for integers by calculating numbers of the same place value.
- Teach pupils how to express the size of fractional parts that do not reach the unit quantity using decimals.
- Teach pupils how to view, in a relative way, numbers expressed with decimals based on 0.1, etc.
- Teach pupils how to express decimals on a number line and read decimals displayed on a number line.
- Teach pupils how to add and subtract decimals in simple cases.
- Have pupils understand the meaning of decimals and how to represent them.
- Have pupils understand decimal addition and subtraction.

3. Regarding Course Units

Up until this point, what the pupils have learned about amounts consisted of clarifying units and learning that the number of such units that can be expressed with an integer. Here, however, pupils will learn to estimate the size of amounts smaller than units, i.e., fractions, and how to express these.

To express the size of a quantity that is smaller than a unit, one uses decimals and fractions. Here, however, the division number differs from arbitrary fractions, with the unit being divided into 10 equal parts, and decimals that can be combined in groups of 10 for a decimal scaling position (decimal system) are taken up.

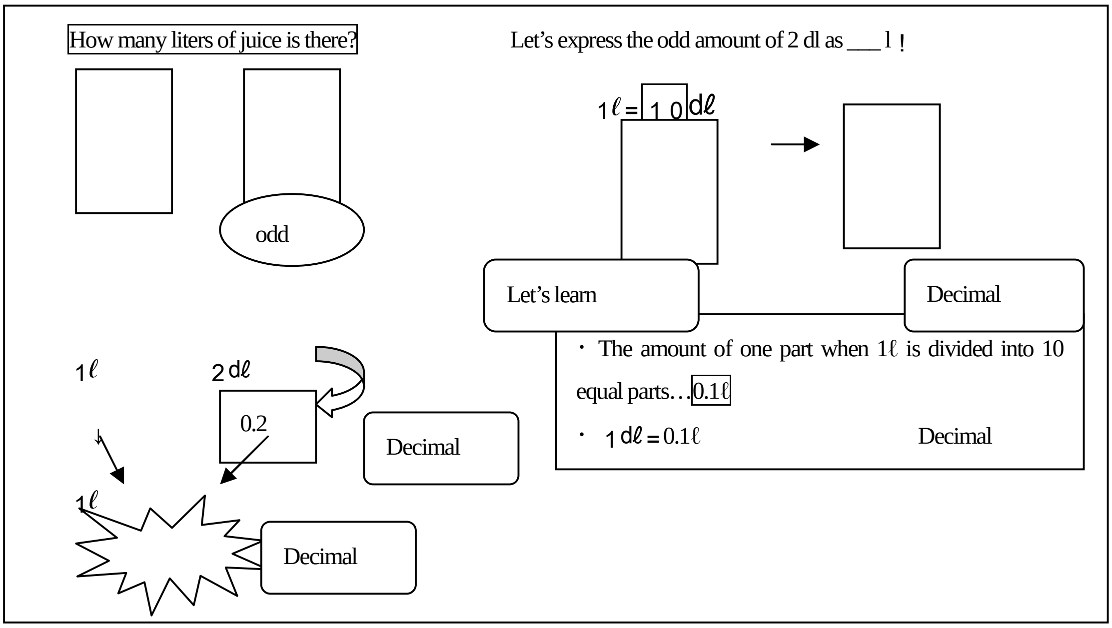
Fourth-year pupils deal with decimals through three course studies, namely, (1) the expression of fraction size, (2) the decimal system and (3) the calculation of decimal addition and subtraction on paper. This class period will introduce decimals as described in (1) and will cover amounts measured in liters. By using a new unit created by dividing 1 l, which is the unit quantity, into 10 equal parts, we will have pupils work on the size of fractions.

In this class period, by having children engage in various activities, such as using colored water to transfer an amount that is less than 1 l into a graduated container by hand while visually checking the operation, we will help them understand that odd amounts can be expressed with decimals.

4. Teaching Plan (8 Hours)

Subunit	Duration (hrs.)		Teaching Contents
1. Expression of fraction size	2	1	Using decimals to express the size of a fraction that does not reach the unit quantity
		1	The ability to express the size of fractions using decimals, even in the case of length (cm)
2. Decimal system	3	1	・ Number line display of decimals ・ Meaning of the term *one decimal place* and the decimal scaling position of decimals
		1	Relative size of decimals and structure and magnitude of numbers
		1	Addition and subtraction of decimals in simple cases
3. Addition and subtraction of decimals	2	1	Addition of decimals by hand up to one decimal place
		1	Subtraction of decimals up to one decimal place
4. Practice and review	1		Review of this class period and practice

5. Blackboard Plan



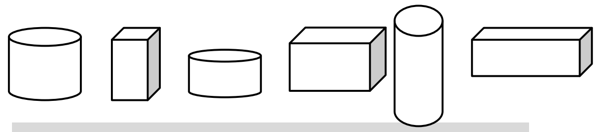
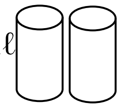
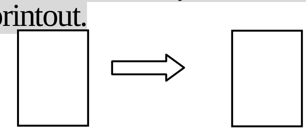
(2) Expansion of this class period (1/8 hr)

6. Lesson of This Class

(1) Course aim

Understand that decimals are used to measure amounts less than the unit quantity.

(2) Course development (1/8 of the class time)

<table>
<tr><th>Pupil Activities</th><th>Role of the Teacher</th></tr>
<tr><td>

○What is the approximate amount contained?
1ℓ  Odd amount○How much is the odd amount?
About 2 dℓ
Less than half of 1ℓ
Estimate: Total amount of juice is 1 ℓ and ○dℓ.
○What would you need to know the exact amount?
Let's use a graduated container to obtain the amount.
Total juice is 1 ℓ and 2 dℓ!!
How many □ℓ of juice is there in all?
Pupils realize that 2 dℓ (odd amount) can be expressed in ℓ.
Let's express how many l 2 dl (odd amount) are.
○Let's accurately write 2 dl in the left box on the printout.

Pupils realize that, similar to dl, l can be divided into 10 equal parts.
Pupils learn that the amount corresponding to one of the 10 equal parts that 1 l has been divided into is 0.1 l (1 dl = 0.1 l).
Amount of odd amount: 2 dℓ = 0.2ℓ ! !
</td><td>
・ Beforehand, arrange the desks to make six groups
Have the children estimate the amount.
・ Explain the measurement method (1 l container without graduations)
・ Walk around the desks and check how the children are doing.
Note: Ensure that the children work carefully without spilling anything.
・ Define the term odd amount: This is an amount that is less than 1 l (unit quantity).
・ Have the children estimate odd amounts using dl, which they learned in their 3rd year.
・ Distribute graduated containers.
・ Explain that regardless of their different shapes, each container holds the same amount.
・ Talk about the benefit of converting 1 l and 2 dl into ___ l.
・ Distribute printouts. Have the pupils review what they have learned up to that point.
・ Confirm that 1 l = 10 dl.
・ Walk around the desks and check how the children are doing.
・ For children who do not understand the concept of 10 equal parts, suggest that the height of the box is 10 cm.
・ Explain using the figures (boxes). Do not make detailed reference to addition.
</td></tr>
</table>

○How much is 1 l plus 0.2 l? 1.2ℓ of juice in all Review The size of the odd amount can be expressed with decimals. Input of impressions, note regarding the next class period	

DEVELOPING MATHEMATICAL THINKING IN A PRIMARY MATHEMATICS CLASSROOM THROUGH LESSON STUDY: AN EXPLORATORY STUDY

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Universiti Sains Malaysia, Malaysia

Abstract

This paper discusses an exploratory study that aims to develop mathematical thinking in a primary mathematics lesson. Although mathematical thinking is one of the significant components of Malaysian school mathematics curriculum, it was not explicitly implemented in many Malaysian schools due to time constraints and mathematics teachers' lack of understanding and awareness about mathematical thinking. In views of the importance of mathematical thinking, it was set up as one of the goals of a Lesson Study group existing in a Chinese primary school. Two lesson study cycles were carried out with a result of two mathematics lessons planned and observed. Five mathematics teachers participated in the study. Preliminary analysis shows that these mathematics teachers espoused that it was much easier to learn new teaching ideas such as developing mathematical thinking through Lesson Study collaboration. Initially, many teachers did not understand fully what mathematical thinking is and how to help pupils to develop this kind of thinking. After two lesson study cycles, these teachers have gained much more understanding and confidence in developing mathematics lessons that promote mathematical thinking. Nevertheless, time constraint and heavy workload remain their two main challenges to integrate any new teaching ideas and strategies.

Introduction

This paper discusses an exploratory study that attempts to develop mathematical thinking in a primary mathematics classroom through Lesson Study collaboration. A document analysis of the Malaysian primary and secondary mathematics curricula done earlier (Lim & Hwa, 2006) indicates that promoting mathematical thinking among Malaysian pupils is an intended goal but it was not explicitly spelled out in the syllabus. A literature search of local studies signify the need to have much more empirical study that focus on promoting mathematical thinking in the Malaysian classroom. Informal discussion with school mathematics teachers displayed that many mathematics teachers agreed to the importance of mathematical thinking and would like to promote mathematical thinking in their classrooms. But they are usually constrained by several issues and challenges such as (i) lack of clear understanding of mathematical thinking; (ii) lack of appropriate assessment tool that measure mathematical thinking and (iii) lack of know-how to promote mathematical thinking (Lim & Hwa, 2006).

In view of the importance of mathematical thinking and the potential of Lesson Study collaboration, an attempt was made to develop mathematical thinking as a goal of an existing Lesson Study group in a Chinese primary school. This Chinese Primary School situated in the centre of an urban area. It is a mini-size school consists of one headmistress, one male teacher, ten female teachers, and 136 pupils. There are only 6 classes with one class for each grade. The Lesson Study group of this school consisted of 8 mathematics teachers were set up since January 2006. They have gone through three lesson study cycles in the year 2006, with the aim of enhancing mathematics teachers' content knowledge and their confidence in teaching mathematics in English language (detailed report see Goh Siew Ching, 2007).

In the following section, I will first explore the teachers' understanding of mathematical thinking; follow by a brief description of the exploratory study. This study comprises of three stages: (a) an introductory workshop on mathematical thinking; (b) first lesson study cycle; and (c) second lesson study cycle. To highlight how these teachers' attempt to develop pupils' mathematical thinking, parts of the two lesson plans collaboratively designed by the Lesson Study group will be used to elaborate together with video clips of the lessons observed.

Teachers' perceptions of mathematical thinking

To elicit mathematics teachers' understanding of mathematical thinking, a brief questionnaire was given to the 6 mathematics teachers and 5 non-mathematics teachers who attended the workshop. Analysis of their response show that majority of these teachers were not sure if they were ready to promote mathematical thinking in the classroom. The main reason was "teachers are not given enough resources to promote mathematical thinking in the classroom". All except two did not answer the question, "Are Malaysian teachers promoting mathematical thinking in the classroom?" The two who answered were also not sure "because they [mathematics teachers] merely convey the knowledge of doing or solving the problems of mathematics."

Out of the 11 teachers, two of them agreed that they understand what mathematical thinking is, two disagreed while others were not sure. Consequently, only two of them agreed that they know how to promote mathematical thinking in the classroom. To these teachers, mathematical thinking refers mainly to problem solving, involve creative and logical thinking, and require skills such as reasoning, analyzing and the use of mathematical symbols. One mathematics teacher believed that she has been incorporated mathematical thinking in her daily teaching although she did not explicitly mention it in class. For her, asking a lot of "why" questions and giving pupils a variety of questions to solve are ways of promoting mathematical thinking.

An exploratory study to promote mathematical thinking

In view of the importance of mathematical thinking and the lack of proper understanding of mathematical thinking among teachers, an exploratory study was proposed to promote mathematical thinking among mathematics teachers through Lesson Study collaboration.

An introductory workshop on mathematical thinking

On March 9, 2007, all the 11 teachers attended an introductory workshop on mathematical thinking. The main aim of the workshop was to expose these teachers to the concept of mathematical thinking and to propose some possible strategies to promote mathematical thinking in the classroom. These teachers were shown a videotaped Japanese classroom lesson of a Grade 4 mathematics topic on “prime and composite number”. Before showing the video, the teachers were given the same classroom activity to experience. Ten cards of different designs were arranged in a specific way. Teachers were asked to observe the order of the designs and determine what the patterns or order represent. They were then asked to identify the rules and using these rules to arrange the successive two cards. The teachers seemed to enjoy this activity and some of them were able to come out with certain kind of rules.

Later, the teachers were shown the video lesson and asked to list out the characteristics of mathematical thinking that they observed in the lesson. The following list was the outcome:

- Activity based
- Pupil centred, active pupil participation
- Justifying, reasoning, argue, debating
- Extrapolating, extend to new situations
- Generalizing, evaluating
- Decision making
- Positive attitude – willing and eager to try
- Logical thinking, creative thinking etc

Based on the list, the teachers were encouraged to plan a mathematics lesson that promotes mathematical thinking through their Lesson Study group collaboration.

Teachers were encouraged to write their reflection after the workshop. Some teachers reported in their written journals that they have been practicing some of the above characteristics of mathematical thinking in their daily class teaching. However, many of them were not aware that these were elements of mathematical thinking. They espoused that they were keen to plan out a mathematics lesson that will help to develop mathematical thinking.

First Lesson Study cycle (22 March-27 April 2007)

Five mathematics teachers participated the first Lesson Study cycle. The topic chosen was “percentage”. See Appendix I for a detail lesson plan. In this cycle, the teachers met four times: 3 meetings for discussion on lesson planning and one for teaching observation followed with reflection and discussion.

Second Lesson Study cycle (13 June-16 July 2007)

In the second Lesson Study cycle, the same five mathematics teachers participated. The topic chosen was “Time” for Grade 4 class. See Appendix II for a detail lesson plan. In this cycle, the teachers met five times: 4 meetings for discussion on lesson planning and one for teaching observation followed with reflection and discussion.

General outline of the lesson

Table 1 displays the general flow or outline of the two lessons. According to the participating teachers, this is also the common format of their normal mathematics lessons. However, small group activities are seldom carried out as it is time consuming. Instead, teachers tend to explain the related mathematical concepts with examples and then give a lot of questions for pupils to practice. Nevertheless, to develop mathematical thinking, they suggested the best way is to promote through small group activity. This is because small group activity will exhibit some characteristics of mathematical thinking, such as active pupil participation, encourage pupils to present and justify their answers, and promote logical and creative thinking.

Table 1: General flow or outline of the lesson

	Lesson 1	Lesson 2
Topic (grade level)	Percentage (Grade 5)	Time (Grade 4)
Learning outcome	Convert proper fraction to percentage	Addition and conversion of time in minutes and hours
Induction set	Represent information in fraction and percentage	Link to pupils' daily life experience: favourite TV programme
Step 1	Small group activity	Small group activity 1: jigsaw puzzle
Step 2	Pupil presentation	Pupil presentation
Step 3	Practice and discussion	Small group activity 2: jigsaw puzzle
Step 4	More practice and discussion	Pupil presentation
Closure	Enrichment exercise-worksheet as homework	Enrichment exercise-worksheet as homework

Developing mathematical thinking in Lesson 1

Pupils were divided into four groups. Each group was given 3 cards, labelled as M, S and E. To stimulate the interest of the pupils, the teacher has creatively linked the cards to M for Monkey; S (Snake) and E (Elephant). Pupils were asked to write down a number between 50-100 for card M; 20-50 for card S and less than 20 for card E.

Without any prior objective of what the number will indicate, pupils simply gave a number that suited the condition. Some wrote 40 over 50 (on card S); some wrote 60 over 100 (on card E). Initially it was planned that these numbers will represent the quiz scores for each group. For example, M stands for Mathematics quiz; S for science quiz and E for English quiz. The mathematics quiz has maximum score of 100; science quiz maximum score is 50 and English quiz maximum score is 20. The pupils were then asked to write their scores in fraction form and later convert to percentage. Finally teacher asked the pupils, "Which group has the best total score to be declared as the winner of the quiz competition? What is the best way to decide?"

This was planned in such a way, so that pupils will need to rationalize [using mathematical thinking] that they have to change the score from fraction form to percentage, so that the three scores can be compared to decide the winner.

However, as reflected by the teacher, Mr L, later in the discussion after teaching observation that he has forgotten this part of the lesson plan. He forgot to ask the pupils to decide which was the best total score. Instead he asked pupils to suggest the best score for each subject.

During the teaching observation, some pupils appeared to be rather unsure about the request of the teacher. One pupil came out to give 19 over 20. But very soon he realized his error and he changed his answer to 20/20. Similarly another pupil wrote 41/100 for mathematics quiz. It was then corrected by his friend to be 100/100. These pupils' answers show that some of them understood that the best score for each subject should be 100%. Nevertheless, it was a pity that the teacher failed to grip the opportunity to encourage more mathematical thinking among pupils, by asking pupils to justify their answers.

Developing mathematical thinking in Lesson 2

Lesson 2 aims to teach the Grade 4 pupils how to add and convert two quantities of time in minutes and hours. The teacher began the lesson by asking pupils' favourite television programme and the amount of time they used to watch these programme per week. This created a cheerful discussion as all pupils were keen to share what were their favourite television programmes. To make the calculation simple, the teacher limited the number of programme to only one per day. As there was no programme on Wednesday, a total of 6 programmes were watched per week. Since each programme was shown for 30 minutes, a total of 6 x 30 minutes which equal to 180 minutes or 3 hours was the total time of watching. This was a direct and simple calculation sum for the pupils. However, to promote mathematical thinking, the teacher challenged the pupils to suggest alternative methods. One girl proposed multiple additions. She then demonstrated her method in front of the class (see Figure 1). She added 30 minutes for six times and yielded the same answer of 180 minutes in total. This is an example of mathematical thinking because pupils were encouraged to show variation in methods of solving.

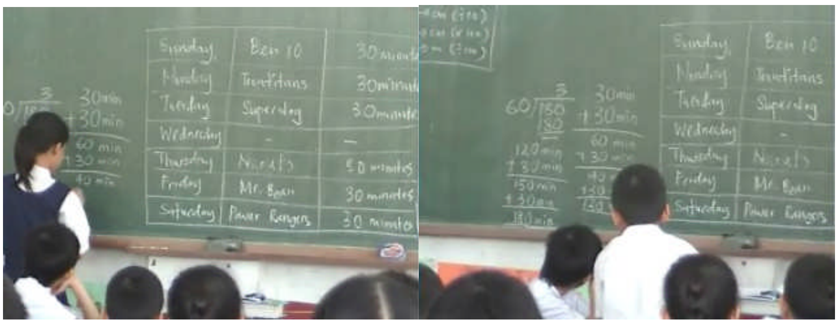

Figure 1: pupil show alternative method

In the second part of lesson 2, pupils were divided into 6 groups. Each group was given an envelope which contained two sets of question. Pupils were encouraged to discuss in group and to match every sheet of paper given to form a correct set of mathematical relationship. For example, match 3 sheets of paper as “45minutes + 50 minutes = 95 minutes”; or “35 minutes + 28 minutes = 1 hour 3 minutes”. All pupils were observed to participate actively and keenly in the given activity. Later, each group presented their solutions to the class. One pupil from each group was also asked to demonstrate their method of solving on the board.

To promote mathematical thinking, the teacher deliberately asked a lot of “why” questions to her pupils. For example, a girl pupil subtract directly 120 minutes from the total sum (see Figure 2), instead of the usual method of divide by 60 minutes. The teacher asked her to justify and the girl was able to explain that 120 minutes equals to 2 hours.

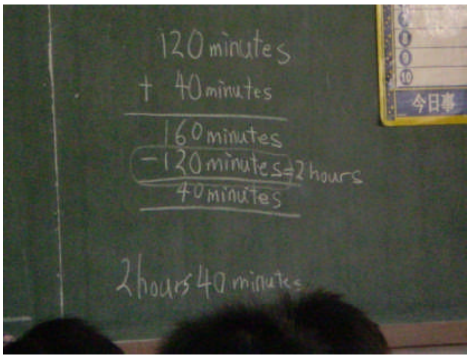

Figure 2

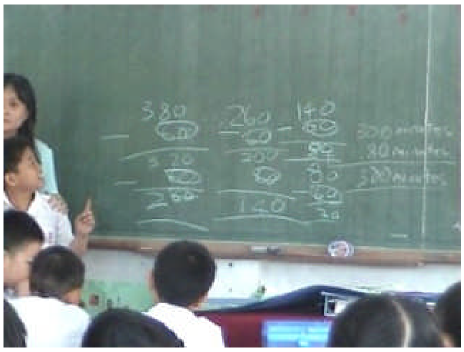

Figure 3

Similar to the first part of lesson 2, the teacher challenged frequently her pupils for alternative methods. For example, in relation to the equation: “300 minutes + 80 minutes = 6 hours 20 minutes”, both pupils displayed the same method of solving as “300 + 80 = 380” and then divide 380 minutes by 60 minutes to give the answer of 6 hours and 20 minutes. So, the teacher challenged her pupils, “Besides divide by 60, is there any other method of getting the answer?” One boy proposed, “minus!” The boy was then asked to demonstrate his method to the class. He displayed how to solve by multiple subtractions (Figure 3).

Taking this opportunity, the teacher also extended pupils’ mathematical thinking to new situations. The following dialogue demonstrates this point:

> Teacher (T): Why do you circle all the 60 and 60?
>
> Pupil (P): Because 60 minutes is one hour.
>
> T: So you circle how many 60s here?
>
> P: 6
>
> T: it means how many hours?
>
> P: 6 hours

……

The teacher also took this opportunity to ask the whole class.

T: so if take away four 60s means how many hours?

P (choral answer): 4 hours

T: how about eight 60s?

P: (choral answer) : 8 hours

Hence, it was observed that the teacher in lesson 2 was working very hard to incorporate mathematical thinking in her lesson. One also noticed that she was code-switching (using either English or Mandarin) to explain and to give instruction so as to ensure that all her pupils understand her teaching. The pupils of this class are made up of three races: Chinese, Indian and Malay. The majority of them do not understand English language very well.

Teachers' reflection

Immediately after each teaching lesson, all the five teachers and the researcher gathered to reflect and discuss. As part of the lesson study process, teachers were also encouraged to write out their reflections in their journals after every discussion and teaching observation. They were allowed to write using any language that they were comfortable with. Out of the five teachers, two of them wrote using English language, two wrote in Mandarin and one wrote in Malay language.

Teachers' Reflection on Lesson 1

The teacher who taught Lesson 1, Mr L expressed that he was rather nervous at the beginning because he was trying to recall and to follow what was planned in the lesson plan. He rated himself as 50% successfully achieved the objectives of the lesson. He was rather happy that even the 4 weakest students in his class seemed to pay attention today. He admitted that he changed what was planned in the lesson plan after the induction set.

The four teachers who observed lesson 1 expressed positive support and comments to Mr L. They contented that Mr L has clear and loud voice, very good rapport with his students, confident, patient and experienced. They also praised each other for preparing colourful power point presentation and worksheets.

One teacher, Ms S pointed that the instruction given by Mr L was rather confused. She saw many pupils did not know how to proceed, and she was rather worried at that time. Consequently, another teacher Ms M proposed that Mr L could have asked the pupils to solve based on one subject at a time and not all three subjects at the same time. Likewise, another teacher, Ms K reflected on herself that given that situation, she would quickly give examples and show to her pupils how to solve them. She was amazed that Mr L was very patient and waited patiently for his pupils to explore and to find out the answers by themselves.

When asked if they have incorporated mathematical thinking in that lesson, they all agreed that they have attained to a small extent. For example, when the teacher asked, "what will be the percentage if there are only 10 questions?" This kind of question encouraged pupils to extend their understanding to solve another kind of questions or there is variation of questions. However, due to time constraint and the pupils' ability, they found it difficult to integrate much mathematical thinking in the lesson.

Nevertheless, when challenged to suggest other possible ways of integrating mathematical thinking in this lesson, they suggested the best way as asking a lot of "why" questions. For examples:

"why must be divided by 100 to get the percentage?"

"why converting from fraction to percentage, we use multiplication? But converting from percentage to fraction, we use division?"

"Why do we need to score full mark to be the winner?"

Another suggestion was encouraging pupils to give alternative methods of solving.

Teachers' Reflection on Lesson 2

Teacher who taught lesson 2 was Ms M. On reflection, she acknowledged that she did not follow the lesson plan strictly. She did not manage to cover all parts of the lesson because she believes that, "if pupils could not understand, there is not point to go on." Due to short of time, she changed the last part of the lesson to ask pupils to continue the following day. For her, today's lesson was not of any special but as what she normally did in class. However, her colleagues who were observing Lesson 2 felt that the class atmosphere was very lively and pupils seemed to enjoy the activity. All teachers were amazed with the number of TV programme and the familiarity of the pupils about these programmes.

Mr L observed that some pupils were able to explain the alternative methods that they suggested, this shows that they were thinking mathematically. He found some pupils were arguing among themselves when they were doing the matching activity. Some pupils used trial and error, some started to write down and calculate. Most pupils seemed engaged and enjoyed themselves. Ms K and Ms S echoed that they were a bit worried that the pupils could complete the matching activity successfully. This was because they have attempted to solve the problems while preparing the activity. It took them quite some times to find a match for one of the questions. They were very happy to see that all pupils could find the answers correctly.

Ms C gave some suggestions for improving the teaching such as pasting the questions on the board so that every pupil can refer to the question. She also suggested that besides the multiple additions (30 + 30 + …) and multiplication (30x6), another way is grouping of 30 + 30 become 1 hour, so 3 groups of 30+30 become 3 hours.

All the teachers agreed that although Lesson 2 appeared simple and easy, teacher Ms M has managed to incorporate mathematical thinking in the lesson. The teacher has asked a lot of "why" questions and has always encouraged pupils to suggest alternative methods of solving. She also encouraged pupils to present their solutions in front of the class.

Teachers' Reflection on Lesson Study

All teachers agreed that participating in lesson study gained them a lot of new ideas and new experiences. They felt better collegial collaboration with their colleagues. However, in spite of the benefits, they felt lesson study was a challenging task. They lamented that each lesson plan using the lesson study cycle required at least 3 to 4 weeks to be completed. In view of the present school system, they were overloaded with tons of paper works besides teaching load. They were over-stressed and rather reluctant to continue lesson study process. These grouses were also reflected in their journal writing. This shows that "time" remains the biggest challenge to the sustainability of lesson study process.

Teachers' Reflection on mathematical thinking

After the two cycles of lesson study, I discussed with the teachers about their understanding and importance of mathematical thinking. The school principal also joined us for the teaching observation of Lesson 2 and the reflection and discussion after that.

Ms C commented that she used to promote mathematical thinking in her normal class, such as variation in difficulty level (from easy to difficult), variation in types of question and variation in methods. However, she was not aware about the term, mathematical thinking. She believes that it is pertinent to encourage pupils to think mathematically. Mathematical thinking should be an important part of mathematics learning.

Mr L supported Ms C's comments about the importance of mathematical thinking. He remarked that mathematics lessons that involve activities that promote mathematical thinking appear more lively and enjoyable. By encouraging pupils to use various kinds of methods will make them more flexible in thinking. This might enhance their adaptability to daily life and future career. All the other teachers also agreed that the normal mathematics lessons are usually very boring and inflexible [死扳]. Pupils are usually asked to follow exactly what the teacher taught. Hence, mathematics lessons should include activities that promote mathematical thinking. The school principal especially agreed that it will be ideal if every mathematics lesson can help to develop pupils' mathematical thinking.

However, time remains the biggest challenge for these teachers. They lamented there were too much workload and documents that they have to prepare daily. Mathematics lessons that promote mathematical thinking usually take time to prepare and to engage pupils to participate. In addition, with the present school system that emphasis on examination, teachers and pupils are forced rushing to finish the syllabus, and to ensure pupils are prepared for examinations. Hence, it is too challenging and stressful to incorporate mathematical thinking in every mathematics lesson unless there is reform in the present school system, examination culture and emphasis of mathematical thinking.

Conclusion

This paper reported an exploratory study that aimed to promote mathematical thinking among pupils of a Chinese primary school in Malaysia. Even though all the five mathematics teachers participated in this study were familiar with lesson study process, they were not clear how to help pupils develop mathematical thinking. For these teachers, several ways of promoting mathematical thinking are (i) ask a lot of "why" questions; (ii) encourage alternative methods of solving; (iii) variation in types of question, so that pupils are encouraged to extend their knowledge to apply to new areas.

These teachers agreed to the importance of mathematical thinking and were keen to promote it. However, they remain sceptical about the practicality and feasibility of this project. They lamented the biggest challenge is time factor. They consider a mathematics lesson that promotes mathematical thinking to be always time consuming and effort driven. Nowadays all teachers are overloaded with both teaching and non-teaching duties. They are always stressed by the school authority and pupils' parents to complete syllabus in time and to ensure their pupils excel in public examinations.

In brief, the experience of this exploratory study implies that it remains a big challenge to promote mathematical thinking in Malaysian schools. Several hindrance are (i) school culture; (ii) teachers' attitude and commitment; (iii) teachers' workload; (iv) exam-oriented culture and (v) assessment system. Unless there are efforts to reduce these hindrance, or else the road to promote mathematical thinking in Malaysian mathematics classroom seems to be still far-fetched.

Acknowledgement

This study was made possible with the cooperation, sacrifice in time and effort of Ms Goh Siew Ching and her school principal and colleagues, as well as the pupils of the two classes. Special thanks also to my two postgraduate students, Hwa Tee Yong and Gan We Ling in helping to video tape the lessons.

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Appendix I: **Lesson Plan 1**

Subject : Mathematics

Year : Year 5

Learning area : Percentage

Sub-topic : (a) Convert proper fractions to percentages.
(b) Convert percentages to fractions.

Duration : 60 minutes

Resources : Blackboard, manila cards, marker pens, cardboards, hundred square paper, LCD, laptop

Key words : Percentage, symbol, percent, hundredths, hundreds squares, parts, convert, fraction, denominator, numerator, equivalent, simplest form.

Learning Objective : Pupils will be able to understand and use percentage.

Learning Outcomes : Pupils will be able to
1. Convert proper fractions to percentages.
2. Convert percentages to fractions in its simplest form.

Previous knowledge: Pupils have already learnt the name and the symbol for percentage.

Values : Self-reliance, logical thinking, mathematical thinking, cooperative, bravery, gratitude, careful, helpful.

Appendix II: **Lesson Plan 2**

Subject : Mathematics

Year : Year 4

Learning area : Unit 5 Time

Sub-topic : Basic operations involving with time:
Add minutes with answers in hours and minutes

Duration : 60 minutes

Resources : Blackboard, manila cards, marker pens, cardboards

Key words : Convert, relationship, involving

Learning Objective : Pupils will be able to do the basic operation involving time.

Learning Outcomes : Pupils will be able to

3. Add minutes with answers in hours and minutes.
4. Convert units of time involving hours and minutes.

Previous knowledge: Pupils have already learnt time in hours and minutes and converting units of time involving hours and minutes.

Values : Logical thinking, mathematical thinking, cooperative, bravery, honesty, careful, helpful.

Step	Content	Activities		Remarks
		Teacher	Student	
Set induction (± 1 minutes)	Asking questions related to their daily life.	T: What is your favourite TV programme?	Various answers will be given by the pupils .	Pupils listen and respond.
Development 1 (± 9min)	Variation of questions .	Teacher asks the following questions and led the pupils to answer. T: So, how many minutes you spend to watch your favourite programmes on Monday? T: Tuesday? Wednesday?..... T: How much time do you spend on watching TV programs in a week? Teacher will draw a table concerned on	Various answers will be given by the pupils Pupils find the duration of the time spent for TV programmes on each day and the total time spend in a week.	Pupils listen and respond.

		the blackboard and asks the pupils to find the duration of the time spent on TV programmes each day and the total of time spent in a week. Teacher also remind the pupils the moral value behind it, i.e. don’t spend too much time on TV programmes, but instead have to choose the good programmes.		
Development 2 ($\pm$ 46min)	Jigsaw puzzle: two sets of questions i) Easier Questions: Purple, pink, Green cards. (20 minutes) ii) Difficult Questions : Blue, yellow, orange cards (20 minutes)	1st round: Easier 1. Divide the pupils into 6 groups. Each group consists of 4 or 5 pupils. 2. One representative of each group comes forward to get an envelope. 3. Inside each envelope, there are 2 pairs of questions. 4. Every pupil in each group think, discuss and to match the correct pairs of questions and answers so as to finish the task. 5. Teacher will then ask the pupils to come out to explain how they get the answers and discusses with the pupils.	Pupils cooperate to find the correct pairs of questions and answer and then paste the answer on a manila card in each group. Pupils present their ‘works’ on the blackboard. Pupils come out to explain how they get their answers.	Pupils discuss and solve the problems.

		2^{nd} round: Difficult 1. Repeat steps 1to 5 as in 1^{st} round.		
Closure (± 4 min)	Conclusion Enrichment	Teacher concludes the lesson. Every pupil will be given a copy of worksheet as homework for enrichment.	Pupils listen and solve the problems.	Worksheets

DEVELOPING MATHEMATICAL THINKING THROUGH LESSON STUDY: INITIAL EFFORTS AND RESULTS

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This paper describes how through lesson study two teachers were made to experience mathematical thinking so that they in turn could create opportunities for their pupils to experience it. In developing the lesson intended for this, they realized the need to deviate from many of their long-held unquestioned practices. This meant trying out what they have not done before in their teaching. Despite some lapses due to their adjusting to the changes in their practices, their actual teaching of the lesson showed that their initial efforts could engage pupils in mathematical thinking. And so, this was a good start.

ANALYZING EXISTING CONDITIONS

Identifying the Usual Practices

The usual components of an elementary mathematics lesson are: drill, review, presentation, developmental activity, fixing skills, generalization, application, and evaluation. In the presentation, a word problem serves as a source of the numbers that are computed by the whole class through the guidance of the teacher during the developmental activity. After the pupils have read the word problem that is expressed in English, it is a standard procedure that they are made to answer guide questions to help them understand and analyse it. Filipino is the native language but English is used to teach mathematics. The guide questions are: (1) What is asked (A)? (2) What is given (G)? (3) What is the word clue/operation to use (O)? (4) What is the number sentence (N)? (6) What is the answer (A)? AGONA implicitly shows how a word problem should be analysed. (Department of Education, 2002).

Determining the Need for a Research Lesson

Last year, the teachers identified that their lesson study goal is to increase pupils' motivation in learning mathematics and to improve their comprehension and analysis of word problems. For this year, they wanted to develop a lesson on solving problems involving subtraction of whole numbers with regrouping because it is a very difficult topic for many pupils. Difficulty is on understanding a foreign language and also on the process of regrouping. To help pupils understand and analyse word problems, teachers code switch and allow the pupils to talk in Filipino or code switch. They teach pupils to look for clue words, words that are associated with a particular operation, so that they will know what operation to use. Examples are deduct, reduce, less, and take away which are associated with subtraction. Relative to forming a number sentence involving subtraction, teachers observe that pupils tend to write the number mentioned first in the word problem as the minuend even if this is actually the subtrahend or the difference. Apparently, the pupils do not understand the word problem. Moreover, they note that in finding the difference of two whole numbers with several digits, there are pupils who always subtract a smaller digit from a bigger digit with the same place value disregarding whether the smaller digit is in the minuend or subtrahend. In effect, they do not use regrouping. The lesson study group has to develop a research lesson that takes into account all these conditions and at the same time develop mathematical thinking among pupils.

Examining the Usual Practices

Since grade 1, pupils have been made to do AGONA as a whole class and it takes time. The teachers were asked to consider other ways to help pupils understand and analyse word problems. These include making them relate what they understand about a it using their own words or asking them what the answer to it is and to explain how they got this. These require pupils to make sense of the word problem. The teachers were also challenged to make their pupils solve the word problem on their own without their guidance. They predicted that many would not be able to do it but a few would. But they were willing to try. They were reminded of the importance of enabling pupils to think on their own.

The teachers were also asked why they teach their pupils to rely on clue words to determine the operation to use in a word problem. It was explained to them that this might not really help pupils' comprehension for they might just look for the clue-words and no longer try to understand the word problem. They were given an example where there was no clue word. (There are 3 500 balls. There are 1 750 balls that are not in the boxes. How many balls are in the boxes?) They were also given a counterexample where the word that is thought to be associated with a particular operation is not so. (The grade 4 classes collected 2 478 stamps. The grade 3 classes collected 1 543 stamps. How many more stamps did the grade 4 classes collected than the grade 3 classes?). Lastly, they were made to realize that word clues are often limited only to the "take away" interpretation of subtraction and disregards its other meanings, namely: additive, comparative, partitive, and incremental (Troutman & Lichtenberg, 1991). Since the teachers were not aware of these other interpretations, they were asked to write word problems on them.

DEVELOPING A LESSON THAT ELICITS MATHEMATICAL THINKING

Using a Framework on Mathematical Thinking

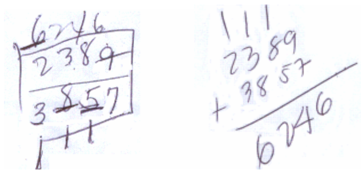
In developing the lesson, the following were considered: teachers must engage in mathematical thinking so that they can elicit this also in pupils; teach mathematics through problem solving to integrate mathematical thinking in the learning of content and enhance pupils' reasoning and communicating skills; encourage pupils to think through questioning; ask pupils to discuss their ideas and formulate their own problems; use non-routine tasks such as open-ended problems to develop mathematical thinking in routine tasks; enable pupils to use their previous knowledge and skills to unfamiliar contexts; connect concepts and procedures; anticipate various pupils' responses; and develop mathematical attitudes like persistence in solving problems and verifying results (APEC Organizing Committee 2006) .

Engaging Teachers in Mathematical thinking

Considering that the procedure in subtracting whole numbers with regrouping had been introduced since grade 3 only with fewer digits, and assuming that pupils could understand English, then this topic would not present anything new from what had already been taken previously. And where would mathematical thinking naturally fit in? So the teachers were probed on what other lessons they had taught so far about subtraction. They had given exercises on finding any of the following, given the other two: difference, subtrahend, or minuend. Building on this, the teachers were then asked in which word problem the pupils would have more opportunity to think - finding one of the following given the other two: difference, subtrahend, or minuend or finding the missing digits in the minuend, subtrahend, and difference which are all given. They were then made to answer the following:

Mr. Jose saves money for his house repair. The repair costs P _246. He has already saved P238_. So he still needs to save P3_ _7. How much does the house repair cost? How much has Mr. Jose saved already? How much more does he need to save?

Shown below is the work of one teacher who consistently used addition to find the missing digits.



The other teacher used a combination of addition and subtraction. Both of them clearly explained their work. They used the relationship of addition and subtraction, the concept of place value, and the process of regrouping. They appreciated solving this word problem involving missing digits and their using different ways to find them. They remarked that this was the type of word problem that could make their pupils think more. Thus they were asked to make similar word problems using the different meanings of subtraction. It was intended that later they would be given an example of an open-ended problem involving numbers with missing digits.

So the teachers made word problems. They were systematic in their explorations. One placed a blank in each column of the number sentence from right to left and in the subtrahend, then minuend, and the difference. The other placed a blank in each column of the number sentence except in the tens place where he placed a blank each in the minuend and the difference. The blanks were distributed in the minuend, subtrahend, and difference. To his surprise, he got many different possible answers. He asked that if this was the case, then what would the correct answer be. His question provided the opportunity to introduce to the teachers what open-ended problems are. He was told that the problem he made was one example. He said that in previous seminars that he had attended, he had already encountered this term. But it was only then that he understood what it meant. Since the teachers engaged in mathematical thinking, they were better prepared to engage their pupils in this experience, too.

While developing the lesson, the teachers realized that the pupils would need time to solve problems. So they decided not to have a drill and review and to have the problems right away. The problems that the teachers made are shown in the lesson plan. Together, the group thought of possible ways that the pupils might solve them.

Preparing the Task

The task consisted of two problems. Problem 1 was supposed to familiarize pupils with subtracting numbers with missing digits. Problems 1 and 2 were similar because their numbers both had missing digits, and they could be solved in different ways. They were different in that the number sentence in Problem 2 had a column with two missing digits and so was open-ended while in Problem 1 each column of the number sentence had only one missing digit and so was not open-ended. Problem 2 had many different correct answers while Problem 1 had only one. Problem 2 used the comparative meaning of subtraction while Problem 1 used the partitive meaning. In Problem 2 a smaller number came first before a bigger number while in Problem 1, a bigger number came first before a smaller number.

In developing the lesson, it was mentioned that a pattern can be observed as one systematically replaced the two blanks with different digits in the tens column of the number sentence in Problem 2. However, time was not enough to go deeply into this. In the lesson itself, it would be enough that pupils would realize that they could substitute different digits and get different correct answers. This would be the first time that they would be solving a problem like this. The teacher themselves would have to investigate the following: If one systematically substituted the digits 0 to 9 in the minuend, then the digits in the difference would be the same. Why? The higher the digit that was substituted in the minuend, the higher would be the digit in the difference. Why? The higher the digit that was substituted in the difference, the higher would be the digit in the minuend. Why? Would these results still be true if the given digit was no longer 9? Why? Why was it that it was only in the tens place of the minuend and difference that the digits differed? Why were the digits the same for all the other place values? What would be the answers if instead of the digits missing on the minuend and the difference they were missing on the minuend and the subtrahend or the subtrahend and the difference? What if instead of two digits on the same column, two consecutive digits on the same row were missing? These questions show that the task is mathematically rich.

ELICITING MATHEMATICAL THINKING IN PUPILS

Problem 1

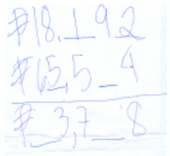
By asking a question that relates to their daily life experience, the teacher got the pupils' interest in the lesson. Being interested can facilitate thinking. By making the pupils read the situation, in particular the numbers that have missing digits, they were made to think because this was the first time that they had something like this. They needed to relate their understanding of place value in order to read correctly each number. Some of them were able to. This was an instance of connecting their existing knowledge to a new context. By making them write the numbers, he focused their attention to the numbers' having missing digits and tried to make them realize the need to find these digits first to answer the questions. Focusing is an important thinking skill.

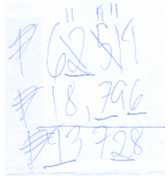
The pupils were not asked to answer AGONA. They were expected to understand what they read and figure out on their own how to answer the questions. Understanding the situation and analysing the relationships implied there without the teacher's guidance required thinking. This was not what they were used to do. Moreover, there were no clue words mentioned on which they could rely. When the teacher asked if they could form a number sentence for the situation, they at first said no. When he raised the question again, they hesitated to answer until he said that there was a problem expressed in what they read. This was the first time that they encountered a problem presented in this way. The number sentences that they had formed before were for those problems that they were familiar with. In what they had now, every number needed was already given but had missing digits. What they had done previously involved two numbers without missing digits and they had to find a third number by applying a given operation on them. So they must be thinking how they would connect what they knew with this new one. If they truly understood what they read, then they could write a correct number sentence that is, represent the relationship symbolically, which is an important thinking skill. In particular, as the video shows, the pupil who was called to write the number sentence was unsure of what he was doing. But with the teacher's coaching, he was assured that what he was doing was right. This action of the teacher was important. He did not give the number sentence himself but gestured approval of the work of a doubting pupil.

After one correct number sentence was given, the teacher asked the class to work in pairs and discuss. Everybody worked and most of those who had seatmates discussed with each other. This discussion provided opportunity for the pupils to think together in determining the digit that should go into each blank. Then, he asked some pupils to explain their work. Explaining one's work required thinking. One had to be able to clearly, completely, and correctly describe one's work in order for others to understand and be convinced that the work was correct. Some explanations were more conceptual than the others. The pupils gave reasons for how they obtained the digits using the inverse relationship of addition and subtraction, the concept of place value, and the process of regrouping. It looked like this problem was easy for most students. Based on the numbers they obtained with completed digits, the teacher asked if they could give another number sentence. He also remarked that if they knew any two numbers, then they could get the third number. The pupils added the subtrahend and the difference and got the minuend. Indirectly, he made them verify if the numbers they got were correct. Implicitly, he also emphasized that addition and subtraction are inverse operations and when two numbers are known, a third number can be known given an operation. Later, the pupils would meet the latter relationship again in the next problem. He also established that now that the numbers did not have missing digits anymore, what they had done before with such numbers could be applied with these ones, too. Recognizing and making all these connections are important mathematical thinking skills. Forming the habit of always checking one's answers is very important. He commented that all their answers were correct although they had different solutions referring to the ways that they used to find the missing digits.

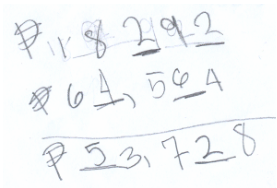
Problem 2

Basically, the teacher's approach was the same as that in the first problem except that he did not ask the pupils to write each number anymore. And so his actions that elicited thinking in them earlier did the same here because in this problem, the numbers were larger and a smaller number was mentioned first before a larger number. When asked to write a number sentence, a pupil wrote this smaller number first then the larger number. Perhaps, because he was not used to being left on his own to understand and analyse a word problem, he did not fully understand it. No one called the attention of the teacher about this nor questioned him or the pupil who wrote it why this was so. Only one pupil realized the error. As shown below, he wrote the bigger number first followed by the smaller number on his paper even before the teacher called their attention.

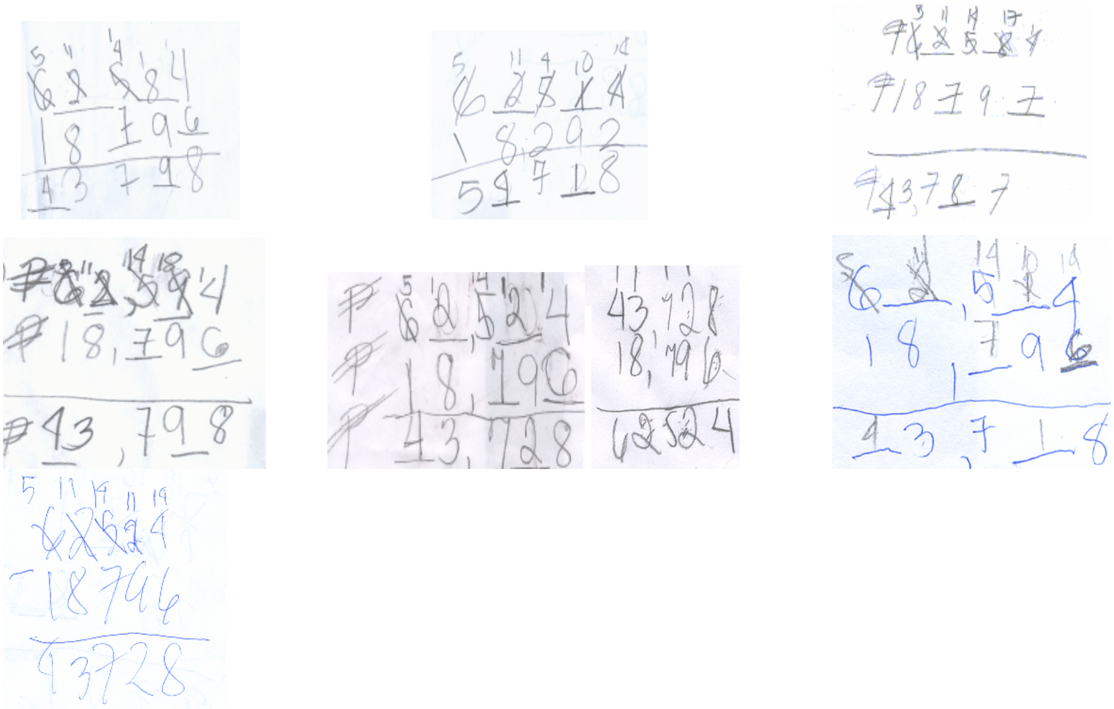




This incident provided an opportunity for pupils to really think. Every one tried to make sense of the number sentence and how to carry out subtraction to get the missing digits. According to an observer, a pupil adjusted the alignment of digits so that 6 is under 8 and so 18 was bigger than 6. Then she affixed 0 at the units-digit of the number above so as to align all the digits correspondingly with those in the number below. Others subtracted the digits in the number below from their corresponding digits in the number above from right to left. When they reached the ten thousands place, they reversed the direction from top to bottom as shown below.



When the teacher realized the mistake, he called the attention of the class by asking which number was bigger. He did not say that the number sentence was incorrect. He wanted the class itself to realize this. And the boy who had corrected the mistake earlier said that the second number should be on top because it was bigger and the number above should be the second number because it was smaller. But apparently, the teacher wanted a conceptual reasoning like this: All the numbers have the same number of digits and the second number has the highest ten thousands digit so it should be written first for the subtraction sentence to be correct. So his question was intended to make pupils think why the number sentence was incorrect. They simply responded “6.” A girl wrote a different number sentence. She must have reasoned that the largest number should be the sum of the two smaller numbers so a number sentence could be one that involved addition. However, he was trying to correct the subtraction sentence, so he did not pursue her answer. Nevertheless, this shows that pupils had different number sentences in mind and that they understood relationships. This indicated that they were thinking. Finally, the number sentence that he expected was written on the board. And the pupils again worked on their seats. Shown below are the works of several pupils. Although some of their answers were incorrect, still these showed that they had an idea that they could substitute different digits to the blanks in the tens place of the minuend or difference. Most of these pupils checked their answers using addition.



Then, the pupils were asked to show and explain their work. It was observed that a pair of pupils was not following the class discussion because they were so preoccupied with finding the missing digits and checking their work. It was not clear in the explanations of those who were called how they determined the missing digit in the tens place of either the minuend or difference, although this must have required thinking from them. The teacher asked if these answers were correct by making them form another number sentence based on their answers. He commented that all their work were correct although their solutions were different.

Presented on the board, were two different correct answers. The teacher called the attention of the class to this by saying: “Look at this number and this number. Is there a difference?” This was to show that for this problem, they had more than one correct answer. In ending the lesson, he asked how many digits were missing in the number sentence in Problem 1. What he actually referred to was the number of missing digits in the tens column. However, this was not clear to the pupils. He also asked for the number of missing digits in the tens place of the number sentence in Problem 2. Although he had no time to elaborate because the class was over, the teacher was attempting to help the pupils recognize a relationship. There was only one correct answer in Problem 1 because there was only one missing digit in the tens column of the number sentence. This meant that two digits were known, so the third digit could be determined using the given operation. In Problem 2, there was more than one correct answer because there were two missing digits in the tens column of the number sentence. This meant that only one digit was known so another digit could be chosen, to get a third number. And there could be more than one choice.

REFLECTING ON NEEDED IMPROVEMENTS

In doing some things for the first time to engage his pupils in mathematical thinking, the teacher showed some good qualities. With commitment, he actively participated in developing an interesting and challenging lesson. In teaching, as much as possible he drew out from the pupils the correct responses by asking them questions. He also gave them enough time for problem solving. But because he was still adjusting, there are still aspects in his teaching that could be improved.

Processing of Pupils' Responses

Proper processing of pupils' responses is very important. The teacher needed to pay more attention to the correctness of their responses given orally or in writing on the board. He missed instances that could have been used to deepen their understanding of mathematics and made them think more deeply. An instance was the misreading of the numbers with missing digits. If he had caught the error and asked them to read correctly, then the number sentence for Problem 2 could have been written correctly. But given what had happened, he should have asked them if they noticed the mistake and that if such happens again next time, then they should ask him. It is important for pupils to have a questioning attitude. Another occasion was the writing on the board of incorrect answers such as the incorrect number sentence for Problem 2. If the error was identified at once, then more time could have been used for exploration in Problem 2 so pupils could possibly observe patterns and make conjectures that are very important in mathematical thinking (Stacey 2006). Another instance also was a boy's giving of the correct answer that the bigger number should be written above the smaller number. He could have probed for the reason for it and used it to lead the class to his expected reasoning. And still another instance was a girl's giving of a number sentence that he did not expect.

He expected to get:

```
  6_ 5_4
- 18 _9_
--------
 _ 3 7_8
```

But the girl wrote:

```
  18_ 9_
+ _37_8
-------
  6_5_4
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This could have been an opportunity to show that even if different number sentences were formed, the different correct answers could still be obtained.

Asking pupils to explain their work is another aspect that needs improvement. The pupils must have done some reasoning to find each missing digit in a number. But when they were asked to explain their work, some simply described how they used subtraction with regrouping in their number sentence. They did not give the reasons on how they found the missing digits in a similar way that the teacher had explained when he worked on the problem in lesson planning. Through questioning, he could have enabled them to explain similarly. This could provide a conceptual meaning to procedures that is important in mathematical thinking.

Giving feedback on the correctness of a pupil's answer by the teacher's probing into his/her response or asking the class to comment on his/her work is also important. Without this, pupils would not know how to move on. This happened to a pupil who was called twice but because he did not know if his first answer was correct, he spent much time checking his second answer that depended on the first. The teacher should also have noted if pupils were careful to use the correct mathematical symbols for their work to be meaningful.

Additionally, board work has to be improved. Writing should be organized and should not be erased to provide a sequential record of what transpired in the lesson and help summarize important points (Wang-Iverson & Yoshida, 2005). The teacher could have guided the pupils in organizing their board work.

Helping Pupils Make Connections

There were several opportunities to emphasize relationships so that pupils could view the lesson coherently. After the missing digits in every number had been determined, the pupils could have been asked to interpret what those numbers were by relating them to the original problem situation. As it was, the pupils simply looked for the missing digits in every number. If the teacher had provided for this, then those who did not put the peso sign would have realized that their answers were meaningless.

Connections could have been made also by focusing on the features of the two problems through asking pupils to compare their similarities and differences. That the second problem could have many different correct answers that could be obtained in many different ways could have stood out if the teacher had asked the pupils to tell what they had observed about the two number sentences for the two problems. He could have done this when the pupils seemed to consider that only one digit could be correctly placed in the blank for the tens digit of the minuend in the second problem.

It is also important for the teacher to involve the whole class in analysing pupils' work. He could have asked them to compare their work and reasoning with those presented on the board and explained in class and to give comments about them.

Making the Most out of Pupils' Work

By consciously spotting the different answers of the pupils while they were working and later calling them to show and explain their work on the board, the teacher could have gathered more different answers and solutions that represented different ways of thinking. From these, he could have asked what they observed about the answers and why they are different but still correct. However, he only called those who raised their hands and so it seemed that the same pupils were answering. He should also have seen if the pupils were really discussing while they were solving the problems.

CONCLUSIONS

With commitment and courage and having engaged in mathematical thinking, the teachers developed and taught a lesson that they had not taught before and in ways that they have done for the first time. They were certainly trying to adjust. They had done the best that they could to introduce problems that would develop pupils' mathematical thinking. Despite certain teaching aspects that need improvements, pupils' responses indicated that they were capable of engaging in mathematical thinking. When the teachers become more at home with their changed practices and engage more in mathematical thinking, possibly the pupils could engage better in mathematical thinking.

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BRIDGES AND OBSTACLES: THE USE OF LESSON STUDY TO IDENTIFY FACTORS THAT ENCOURAGE OR DISCOURAGE MATHEMATICAL THINKING AMONGST PRIMARY SCHOOL STUDENTS

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The first part of the paper describes a study conducted to explore the use of lesson study as a professional development tool. The specific aims of the study and the main stages of the study are described. The second part describes how lesson study helped teachers understand the factors that encourage or discourage mathematical thinking during a lesson. The third part briefly discusses the role of lesson study in enhancing teachers' pedagogical knowledge. The final part outlines a research agenda that employs lesson study to help teachers develop approaches to cultivate mathematical thinking amongst students.

INTRODUCTION

The Singapore mathematics curriculum which focuses on mathematical problem solving was introduced in 1992 and was revised in 2001 and, again, in 2007. Increasingly, the shift with each revision of the curriculum is less emphasis on computational, procedural skills and more emphasis on mathematical thinking. Mathematical thinking is integral in the process of problem solving.

It is, thus, important for teachers to understand the idea of mathematical thinking and how to cultivate it amongst students. However, teachers need to re-examine their own mathematical thinking and their perception of what mathematical thinking is.

Lesson study provides a concrete image and specific situations of mathematical thinking amongst students as they unfold in a classroom. The research lessons provide opportunities to capture the complexities in understanding what mathematical thinking is and the pedagogy associated with its development, that otherwise may not be captured.

THE STUDY

A group of eight teachers in a primary school in Singapore was involved in a six-week lesson study cycle. The aim of the study was to explore the use of lesson study as a professional development tool. In particular, the study reported in this paper focused on two goals. One goal was to enhance the teachers' pedagogy with respect to cultivating mathematical thinking. The other goal was to enhance the teachers' own mathematical thinking and their understanding of mathematical thinking.

In the first session, the teachers were familiarized with the ideas of visualization and generalization as possible aspects of mathematical thinking (Yeap, 2006). The teachers then used a topic (angles) that they were going to teach in the coming weeks to anchor their discussion. The teachers studied the textbooks, workbooks, teachers' guides and other resources that were available including manipulative materials. The discussion culminated in the teachers identifying ideas in the topic of angles that would be a challenge or otherwise for the primary four (grade four) students. The research theme for the research lesson was decided to be helping student construct a visual representation of angles with a focus on a representation that was thought to be challenging for the students. It was thought that students find it difficult to form a visual representation of an unknown angle a when $a + b$ is known (Figure 1).

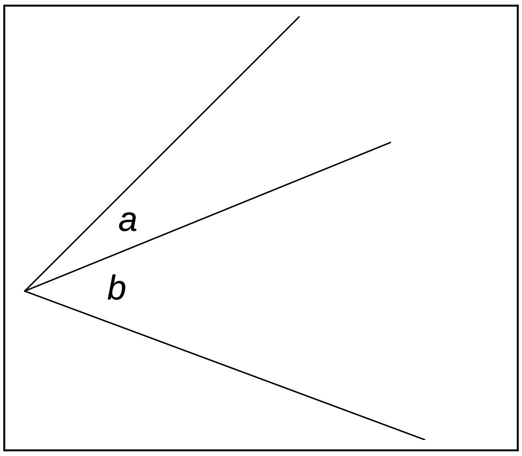

Figure 1: While students are able to tell the angle $a + b$ when angles a and b are given, they find it more challenging to tell an unknown angle when $a + b$ and either a or b are given.

In the second session, the teachers designed the lesson, wrote the lesson plan and made the necessary preparation for the research lesson. They worked out the solutions of the problems they planned to pose to the students and the anticipated students' responses.

As the focus of the lesson study was to develop a possible approach to help students construct a visual representation of angles with a particular emphasis on representations that are considered challenging to the students, the teachers decided to employ the use of concrete materials. Cut-outs of sectors that show 20^o or 30^o were prepared. The use of these cut-outs was, in the opinion of the teachers, helpful in assisting students construct the target mental representations.

The teachers started with the problem of showing different angles using the cut-out pieces. As they solved this problem, they found that the problem was too open. They had to decide on the number of each type of cut-outs to use. They had to decide on how the cut-outs were to be arranged – adjacent to each other, on top of each other or a combination of the two methods. For example, a 20^o piece and a 30^o piece can be placed side by side to show an angle of 50^o. A 20^o piece can also be placed on top of a 30^o piece to show an angle of 10^o. As they solved the problem, they also predicted that students would find the latter challenging, although the use of cut-outs may be useful.

According to the lesson plan, students were to be given a certain number of cut-outs of sectors that show 20^o or 30^o. In this lesson, students were asked to use (a) a 20^o piece and a 30^o piece, and (b) two 20^o pieces and a 30^o piece. Students were required to use these to show other angles.

The research lesson and a post-lesson discussion were conducted in the third session. One of the teachers taught the lesson to a primary four class. Each teacher observed one group of students. The teachers were reminded that they needed to observe students carefully to collect information on student thinking. The next part of this paper focuses on how this research lesson and the information collected helped the teachers identify factors that encouraged or discouraged mathematical thinking during a lesson.

The fourth session focused on revising the lesson plan based on the findings of the research lesson. Subsequently, another teacher taught the lesson to a different primary four class. A post-lesson discussion was again conducted. The final session was spent identifying parts of the lesson plan where mathematical thinking is prominent and delineating teacher actions that are able to stimulate, scaffold, encourage and perpetuate mathematical thinking.

FACTORS THAT ENCOURAGED OR DISCOURAGED MATHEMATICAL THINKING

The first research lesson was made of five main segments. Figure 2 describes the structure of the first research lesson which was 30 minutes long.

Segment	Description	Time
1	The teacher reviewed the idea of angles generally.	00 : 00
2	The teacher helped students understand the task for the lesson using one red (20^o) cut-out and one green (30^o) cut-out.	03 : 10
3	The students worked in groups using two green (30^o) cut-outs and one red (20^o) cut-out.	06 : 55
4	The teacher used two groups' solutions to lead a whole-class discussion.	12 : 00
5	The teacher conducted a general conclusion to the lesson	25 : 50

Figure 2: The structure of the first research lesson

The post-lesson discussion focused on the research theme – to develop a possible approach to help students construct a visual representation of angles with a particular emphasis on representations that are considered challenging to the students. The bulk of the post-lesson discussion was on the factors that encouraged or discouraged mathematical thinking. The following paragraphs are a synthesis of the post-lesson discussion.

The use of the cut-outs was critical in helping some students construct a visual representation of the central ideas of the lesson. This was particularly true in the challenging cases.

Many students did not face difficulty when the cut-outs were placed adjacent to each other. Thus, many students were able to see how a green piece and a red piece could show 50^o readily. The role of the cut-outs differed among different students in this situation. There were students to whom the cut-outs did not matter. They could say how 50^o could be shown without using the cut-outs. These students already had the visual representation of the idea and were making use of it to complete the tasks confidently. Then, there were students who used the cut-outs to strengthen their visual representation. They could say how 50^o could be shown but used the cut-outs to confirm their thinking. Finally, there were students who needed to use the cut-outs to arrive at the conclusion of how 50^o could be shown.

Many students had difficulties when the cut-outs were placed on each other. Thus, not many students were able to see how a green piece and a red piece could show 10^o by placing the red piece on top of the green piece in a certain way. The few students who could still needed the cut-outs to confirm their thinking.

The majority of the students needed the scaffolding provided by the teacher to make the cut-out useful in developing a visual representation of the idea. As the scaffolding was important, the teachers agreed that they needed to be more rigorous in developing the scaffolding questions. This was done for the second research lesson and the positive effects of carefully-constructed scaffolding questions were apparent.

The arrangement for students to work together in groups provided opportunities for students to encounter responses that differed from one’s own. This led to students questioning their peers, seeking clarifications, defending their responses and resolving conflicting views. Such extended engagement with ideas was found to be conducive for mathematical thinking.

The use of the worksheet did not allow for such extended engagement. Answers had to be obtained and recorded promptly. In completing such a worksheet, the students were more eager to have an answer they can record to the teachers’ satisfaction. There was little opportunity for engagement with ideas. It was decided that it would be better not to require students to complete a worksheet where answers had to be obtained and recorded quickly. In the second research lesson, the worksheet was not used. Instead, students were given an individual worksheet at the end of the lesson to consolidate the ideas that they had discussed in the lesson.

While the majority of the lesson was focused on a set of related problems, the first and last segments of the lesson were too general to be useful. General, superficial discussion of ideas did not facilitate mathematical thinking. On the other hand, students working on one problem that a set of solution ranging from obvious ones to challenging ones facilitated mathematical thinking. In the second research lesson, these segments were removed without affecting the main aims of the lesson. The time was used instead to complete the individual worksheet at the end of the lesson.

The problem used in the lesson was open enough to engage students in mathematical thinking. However, the teacher provided the suggestion that the pieces could be placed on each other even before the students had a chance to consider it. This premature direction robbed the students with a chance to make sense of the situation. In the first research lesson, there were students who simply placed the red piece on the green pieces without understanding its significance. This was because the teacher had said that the pieces could overlap. This suggestion was not given in the second research lesson. While fewer groups came up with this method of showing angles independently, these groups need no further help from the teacher in understanding its significance.

The information the teachers collected during the research lesson had resulted in teacher understanding of factors that facilitated mathematical thinking and those that were obstacles to mathematical thinking. Generally, the following was found to be a bridge to mathematical thinking: (a) the use of concrete material to anchor students' thinking, (b) the use of carefully crafted scaffolding questions to help student clear challenging situations a step at a time, and (c) extended engagement with ideas where students encountered different and, sometimes, conflicting views and where they had to question, clarify, justify and defend ideas. The following were found to be obstacles to mathematical thinking: (a) the use of worksheet that required a response to be recorded promptly, (b) the use of closed problems or the conversion of open problems to closed ones by providing directions too early in the problem-solving process.

LESSON STUDY IN DEVELOPING PEDAGOGICAL KNOWLEDGE

The data collected from this study involving eight teachers going through one lesson study cycle in helping teachers develop approaches to cultivate mathematical thinking amongst students allows a brief discussion on the use of lesson study in developing pedagogical knowledge.

In the lesson planning phase, solving the problems themselves allowed teachers to experience mathematical thinking and clarify to themselves what mathematical thinking means. In solving the problem in this study (finding angles that can be shown using a number of cut-outs that show 20^o and 30^o), one teacher very quickly realized the idea that "all multiples of twenty and thirty can be shown", to which another teacher extended when she said "so can all multiples of fifty". The former later included generalizing as an important part of mathematical thinking. Another teacher saw that 20^o can be shown by placing one cut-out on the other. He was made to clarify what he meant and to justify his thinking as several of his colleagues did not understand him. He included defending one's idea as an important part of mathematical thinking. In lesson study, the lesson planning stage included opportunities to reflect and articulate one's thinking in solving the problems selected for the lesson. In individual lesson planning, the reflection and articulation opportunities are left to chance.

In the research lesson phase, observing the students' thinking closely allowed teachers to see mathematical thinking in action. They are also able to see aspects of mathematical thinking that are easy for the students and those which are challenging. Teachers are also able to see instructional strategies that facilitate or inhibit mathematical thinking. In cases where the teachers have the opportunity to revise the lesson plan and conduct a second research lesson, they are able to test their conjectures. The research lesson also shows up instructional strategies that require more careful planning. In this study, the teachers initially did not realize the need to plan the scaffolding questions closely. As a result, the challenging part of the problem (the case of overlap) was not grasped by many students. In the revised lesson plan, the scaffolding questions were carefully crafted. This revised action bore positive effects in the second research lesson. The research lessons, thus, have the twin roles of showing the facilitating or inhibiting effects of instructional strategies including when these strategies are absent or not rigorously designed.

A RESEARCH AGENDA

In Singapore, professional development courses offered by the National Institute of Education are typically in the form of 24-hour courses. A new in-service course in the form of lesson study will be proposed. The structure of the course will be similar to the one described here with an initial session to introduce the lesson study process and a final session to allow teams to share their experience.

The research questions are (1) How do teachers develop their pedagogy in cultivating mathematical thinking amongst primary school students through lesson study? (2) What are the effects of lesson study on the teachers' mathematical thinking, perception of what mathematical thinking is and pedagogical knowledge of cultivating mathematical thinking?

Instruments will be developed to collect data for teachers' mathematical thinking and their perception of what mathematical thinking is. Changes in teachers' pedagogical knowledge will be based on field notes collected during the sessions to study the instructional materials, to plan and revise lesson, to discuss the research lessons and to identify specific points during a lesson where there is significant mathematical thinking and instructional strategies that support it.

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THE VAN HIELE LEVELS OF GEOMETRICAL THOUGH IN AN IN-SERVICE TRAINING SETTING IN SOUTH AFRICA

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ABSTRACT

This short presentation reports on an in-service training programme of Primary School teachers in a mining town in Mpumalanga, one of the nine provinces in South Africa. At this particular session, teachers were placed in a simulated classroom situation where they were exposed to the van Hiele levels of Geometrical thought. This session mainly concentrated on van Hiele level zero (visualisation). Various two dimensional shapes were provided to teachers in groups. The following procedures were followed:

- One teacher would choose a shape, and the rest of the group would then describe the shape
- The groups were asked to classify the shapes according to given properties

The videotape which will be shown, reveals most interesting thinking processes of teachers, which can be used fruitfully in any teaching environment.

In another video clip, an example of intervention that was not conducive for mathematical thinking will be shown, which can be used as a sample for discussion.

USING LESSON STUDY TO CONNECT PROCEDURAL KNOWLEDGE WITH MATHEMATICAL THINKING

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Marian Palumbo, Bernards Township Public Schools, USA

Developing U.S. students' mathematical thinking frequently is an elusive goal. The reasons are varied. Some of them include: 1. teachers' own lack of understanding of mathematics caused in part by an absence of a coherent mathematics curriculum (Schmidt et al., 2002) ; 2. insufficient or no professional development focused on the scope and sequence of mathematics within and across the grades; 3. inadequate knowledge and concrete examples of what mathematical thinking entails for both students and teachers; 4. lack of clear and explicit examples for how to connect students' procedural knowledge with conceptual understanding through mathematical thinking.

To focus APEC (Asia-Pacific Economic Cooperation) member economy specialists' attention on the importance of and approaches to the development of mathematical thinking of both students and teachers, the 2-7 December 2006 APEC lesson study conference in Tokyo/Sapporo, Japan, offered various keynote presentations (Katagiri, March, 2007, Lin, March, 2007, Stacey, March, 2007, Tall, March 2007). The speakers shared their perspectives on approaches to developing mathematical thinking, thus setting the stage for observation and discussion of four lessons, discussion of specialists' papers on mathematical thinking, and preparation for work following the conference. Prior to the end of the conference, the APEC member economy specialists were charged with the task of returning to their country and conducting a lesson study cycle that helped teachers work with their students to develop mathematical thinking skills while working on a specific mathematical concept.

Getting Started

To carry out the assigned task, the U.S. representative to the APEC lesson study conference (Wang-Iverson) invited the mathematics supervisor (Palumbo) at Bernards Township Public Schools (New Jersey) to identify a group of teachers willing to participate in lesson study. Although lesson study has been implemented at various sites across the United States since 1999[1], the team of five grade seven teachers (William Annin Middle School) that agreed to participate in the project was new both to lesson study and to discussing collaboratively how to develop students' mathematical thinking. Palumbo had engaged in lesson study previously with a few high school mathematics teachers, but she had not established a systemic lesson study initiative. The project thus

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[1] Paterson School No. 2 was the first U.S. school to begin lesson study, under the tutelage of teachers from Greenwich Japanese School, a relationship facilitated by Makoto Yoshida (2004).

was two-pronged: introducing these individuals to the purpose, practice, and outcomes of lesson study and facilitating their collaborative work to develop student mathematical thinking through creating and teaching a lesson.

A unique feature of this district is that subject matter teachers at the same grade meet regularly to identify and discuss topics to be covered the following week and to share responsibilities for developing worksheets and homework assignments to be used across the classes. Although regular meetings are common in some districts, rarely do teachers progress at the same pace and share worksheets. What did not occur, prior to their engagement with lesson study, was observing each other's classes and discussing what was observed and what changes needed to be made to foster better student learning. These teachers' lack of opportunity to observe their peers and to be observed in turn is not an uncommon phenomenon across most countries, but U.S. teachers have even fewer opportunities. Internationally, 27% of grade 8 students participating in TIMSS 2003 had teachers who reported they had opportunities to observe colleagues two to three times a month; in the U.S. the number was 11%. Eighty-five percent of U.S. students had teachers reporting they have never observed or been observed by colleagues (Mullis et al. 2004).

During the introduction to lesson study, the teachers identified the characteristics of ideal students vs. the real students they encountered in their classes (see Appendix 1). This list was to serve as the basis for developing a lesson focused on moving students toward more idealistic behavior in learning mathematics. A common trait among students was their focus on simply getting the right answer and moving on to the next task.

Given the common schedule shared by the teachers, they next reviewed the topic they would be covering around the spring dates selected for teaching the lesson. However, after observing some classes on proportions and noting students' tenuous grasp of the concept, the authors suggested the teachers might wish to revisit the concepts of percent and proportion and create a lesson that helped strengthen students' understanding of those topics.

Facilitated by the mathematics supervisor, the teachers developed a lesson study schedule that allowed them to meet weekly or biweekly to plan and develop the lesson. The supervisor suggested the lesson study group use moodle (www.moodle.com), a web-based platform, to document conversations between meetings, which then could be archived. However, this practice was not maintained throughout the lesson study process, as it introduced yet another new undertaking for the teachers in the midst of continuing their regular work of daily teaching.

After a very brief introduction to kyozaikenkyu[2] (Takahashi et al., 2005), the teachers investigated various resources to find problems they wanted to use in their lesson. Through their individual exploration of resources beyond just the textbook, the teachers selected problems that might push student thinking about fractions, percents and proportions. They then reviewed the problems and ranked them. At a subsequent meeting they discussed the merits of the problems selected and agreed upon one problem, which is the first problem presented in the lesson plan (Appendix 2):

Problem: The Carters are buying a new iPod Nano. Three stores have on sale this week the model they want, but they have decided to shop at Ralph's, because they think Ralph's is offering a "double discount." Here are the ads. Did the Carters make a wise decision? Explain.

Radio Shop Original Price $172 Discount: ¼ off	Discount City Original Price: $180 Discount: 30% off	Ralph's Original Price $180 Discount: 10% off with an additional 20% off the discounted price

Identifying the goals was not a simple task; such an approach previously had not been the norm in preparing a lesson. The goals elucidated in the final lesson plan for the Algebra I lesson were focused more on the specific skills rather than on developing students' mathematical thinking:

- **a.** Understand the value for using efficient methods when solving percent problems
- **b.** Compare and contrast the relationship(s) between determining the "part" and "determining the "whole" in a percent problem

Developing and teaching the lesson

The lesson plan evolved over several meetings. One teacher volunteered to work on writing the rationale for choosing the particular lesson problem, another focused on writing the lesson plan itself, and the supervisor and one of the teachers who also taught grade six developed the scope and sequence of concepts taught in the earlier grades (see lesson plan in Appendix 2). However, in the effort to move on to develop the lesson plan,

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[2] "investigation of instructional materials," encompassing not just textbooks, teacher manuals, and mathematics manipulatives, but a wider range of materials, including the course of study (standards), the educational context, learning goals, tools, research and case study publications, lesson plans and reports from lesson study open houses, and ideas gained from research lesson observations. *Kyozaikenkyu* also includes investigation of students' prior knowledge, learning experiences, state of learning and understanding, which makes it possible for teachers to be able to anticipate students' reactions and solutions to the problems students study during the lesson.

the scope and sequence and the accompanying text pages were not studied in detail and discussed by the group in planning the lesson.

Initially, the teachers seemed reluctant to volunteer to teach the lesson, but at the next meeting, they all expressed a desire to teach, as they were interested in having their colleagues observe their students. It was agreed that the lessons would go through two paired iterations in grade 7 classrooms followed by a final iteration in a grade 7 algebra classroom (see table). Two teachers would conduct the first teaching (1-1 and 2-1); to avoid being influenced by the first lesson, the second teacher would not observe the first lesson. This format did not follow the usual lesson study process, where one teacher volunteers to teach the lesson, followed by discussion and revision of the lesson. Whether teaching of the revised lesson takes place varies across lesson study groups.

The structure adopted for this lesson study project provided more opportunities for the teachers to practice their observational skills focused on student thinking and learning. Following the lessons, the team met to share and discuss the data collected and to revise the lesson for the second teaching by two more teachers.

During the first teaching the students simply had been asked to solve the problems on the worksheet. For the second teaching, students received a worksheet that provided room for them to solve the problem in more than one way and to record the time when they finished the problem (see Fig. 1). The time recorded by the students provided useful data for the teachers for scheduling more effectively in the future the amount of time needed by students to complete assigned tasks. The students in these two classes were asked to write down their reflections on a form containing specific questions (see Appendix 2). After the second teaching, the lesson was revised again and re-taught by the teachers in their other classes without observers.

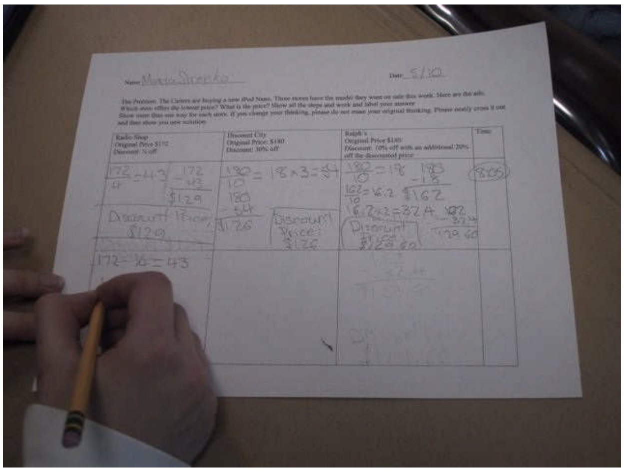

Figure 1. Student worksheet

Table: Teaching sequence for observed lessons

	First teaching	Second teaching	Third teaching
Teacher #1	1-1		
Teacher #2	2-1		
Teacher #3		3-2 (with calculator)	
Teacher #4		4-2 (w/o calculator)	
Teacher #5			5-3 (algebra)

Investigating additional factors

Two teachers used the lesson to investigate how other factors affect student thinking:

1) Calculator usage: One of the teachers during the second teaching of the lesson did not give students calculators to use during the lesson, which provided an opportunity for observers to analyze differences in student work and thinking with/without the use of calculators.

2) Advanced students: One of the teachers who taught a grade 7 algebra class in addition to regular grade 7 mathematics classes further modified the lesson and taught it to the algebra class (final teaching of the lesson). She was able to assess the differences in mathematical thinking between the grade 7 regular mathematics students and her grade 7 algebra students, who were considered more advanced mathematically.

Documentation of student work

During the first teaching of the lesson, some students attempted to solve the problem using proportions but set up the problem incorrectly (see Fig. 2):

25/172 = *x*/100; 30/180 = *x*/100; 10/180 = *x*/100

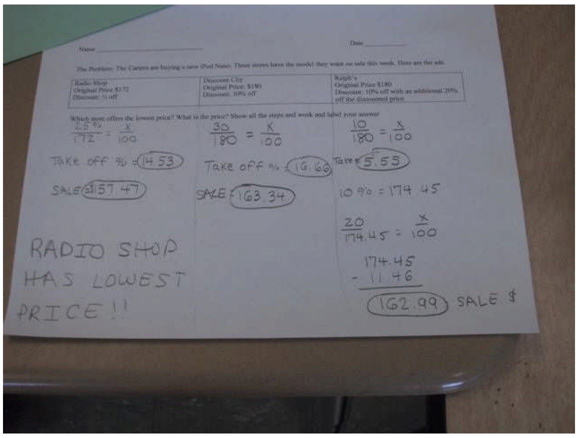

Figure 2: Error using proportions

As a result, students were not in agreement on which discount provided the lowest price. For students who arrived at the correct conclusion that Discount City provided the largest discount, they first calculated the discount and then subtracted the value from the original price:

Given: \$180 = original price; 30% discount

180 x 0.3 = 54
180 – 54 = \$126 = discounted price at Discount City

No students in these two classes solved the problem by directly calculating the fraction or percent of the original price:

180 x 0.7 = \$126 = discounted price at Discount City

In the second iteration of the lesson, when students were asked to solve the problem in more than one way, seven out of 25 students in one class subtracted ¼ from one and found ¾ of \$172 to calculate the discount price at Radio Shop. In the other class, only one student used this method to solve the problem. Most of the other students, in trying to find a different method, moved between multiplying by a fraction and multiplying by the fraction's decimal representation, considering these to be different solution methods. Six of the 44 students in the two classes did not show a second way of solving the problem. These results implied the students may not have been used to being asked to solve problems by more than one method, and some did not understand what it meant to think about tackling the problems in different ways.

Confronting a more challenging problem

During the first teaching of the lesson (1-1 and 2-1), no student was able to solve the additional problem, which asked for the original price of the computer, given the discounted price:

Additional problem: A computer is discounted 20% from its original price because it didn't sell. The store took an additional 30% off the discounted price. Barbara purchased the computer for \$896. What was the original price of the computer?

Three students in one of the classes during the second iteration of the lesson obtained the correct original price: one student had the correct calculator-generated answer but no written record, while the other two students solved the problem using the following steps:

100% - 30% = 70%
896 ÷ 70/100 = 896 x 100/70 = 89600/70 = 1280
100% - 20% = 80%
1280 ÷ 100/80 = \$1600

In using the above steps, these students were able to apply the knowledge used in the earlier problem (subtracting the discount from 100%), but they needed to go one step further to realize that in order to calculate the original price, they needed to divide rather than multiply. Students who were not able to solve the problem correctly did make valiant efforts, trying to apply what they had learned previously. Some set up the proportion formula, *a/b* = *p*/100 (taught earlier in the year by the teacher from the textbook), but then did not know what to do next, demonstrating they remembered but did not understand the formula (Stacey, 2007, p. 45). Students fell into the trap of either multiplying the discounted price by the percent discount ($896 x 0.3) or dividing by the percent discount ($896 ÷ 0.3). Other students multiplied by 0.8 and 0.7. A few students knew to divide by 70% but then divided by 7 and not 0.7. Further analysis and conversation with the students might have helped to determine whether this error is merely computational in nature or reveals a more fundamental problem in moving from percent to decimal notation.

One student arrived at an answer of $949.76 by the following route:

896 x 0.3 = 268.8
268.8 x 0.2 = 53.76
896 + 53.76 = $949.76

The student incorrectly applied the strategy used in the earlier problem (Ralph's store): sequential multiplication. In this case, seeing that $53.76 could not be correct, since the original price had to be greater than the discounted price, s/he then simply added this value to the final discounted price to arrive at the 'original' price. This solution illustrates the student's tenuous grasp of the earlier solution method, leading to an inability to apply it to a different problem.

Another student obtained an answer of $1396.96 using the following method:

896 x 0.3 = 268 4/5
896 + 268 4/5
1164.8 x 2/10 = 252.96
1164 + 232.96 = $1396.96 = original price

In addition to moving between the use of fraction and decimal in solving the problem incorrectly, this student also tried to apply directly what was previously discussed for a different problem to find the discounted price. Perhaps in an effort to compensate for the difference between the two problems, in lieu of subtracting, the student added to arrive at the original price. In this case it would have been useful to ask the student to explain the thinking behind the calculations.

The above two examples illustrate students' readiness to 'push buttons' to arrive at an answer but an inability to evaluate the work to make sense of the calculations. In developing mathematical thinking, students need to learn to slow down and to be taught explicitly how to engage in metacognition, scrutinizing one's own thinking. This lesson study cycle revealed the need to help students move beyond simply applying algorithms without considering whether they make sense for solving the specific problems.

In the advanced class there were no computational errors. Twelve of the 16 students solved the iPod problem by first calculating the discount and then subtracting it from the original price; the remaining four students directly calculated the discounted price for the iPod problem. These same four students calculated the price at Ralph's using a two-step process: first calculating the 10% discount followed by the 20% discount.

One student in this advanced class initially calculated the answer for Ralph's by the following method:

180 x 0.1 = 18
180 – 18 = 162
162 x 0.2 = 32.4
162 – 32.4 = $129.60

From this solution, she then was able to reduce the steps to one equation:

x = (180)(0.9)(0.8) (see Fig. 3).

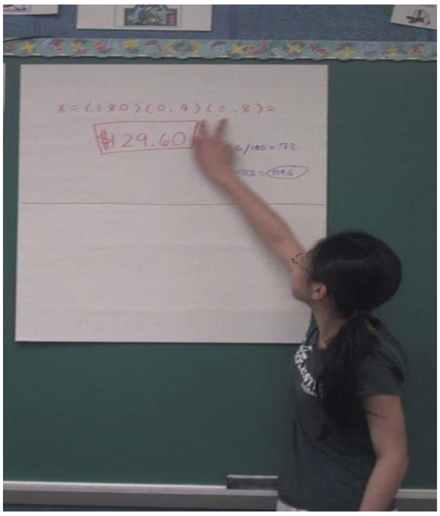

Figure 3: Expressing the iPod problem in one step.

After the student presentations, the teacher summarized the approach to using a one-step equation to finding the cost of the iPod at Ralph's. She then asked the students to solve the computer problem. The majority of students did not solve the problem correctly; in that time period they were not able to transfer what they had learned from the previous problem. A few students were able to solve the problem by first dividing by 0.7 and then dividing by 0.8, which was a solution anticipated by the teacher (see lesson plan in Appendix 2). One student reduced the equation to:

Original price = 896 ÷ 0.56 = $1600

For most of the students, using the more efficient method of solving for a double discount presented a new way of thinking about the problem. In hindsight, perhaps they needed to have first solved a problem asking for the original price after a single discount and then move on to the computer problem with its two discounts[3].

Student discourse

Students worked in pairs in all the classes. Three patterns of behavior[4] were observed: 1. although asked to work with a partner, students worked silently and individually; 2. one student immediately took charge and told the other student what to do; 3. the two students worked as a team, discussing their answers as they worked.

Two students in particular during the third teaching (5-3) carried out a prolonged discussion of their answer of $14,933.30 for the computer problem, which they had obtained by dividing the sale price of $896 by 0.3 and then by 0.2. The original price they obtained seemed too high to them, but in checking it, using the same decimals, they came up with the same number. They concluded the answer had to be right, despite feeling perplexed by the large number. Neither student questioned the validity of their thinking; they simply checked their calculation without considering that perhaps they were using the wrong numbers.

Another pair of students in class 4-2 engaged in a debate over what one student had written for the proportion they had set up. The second student insisted the first student's work was wrong, while the first student replied that what she had written was correct. The first student finally understood the source of the second student's disagreement and said that the proportion she had set up was correct, but that she simply had written it as $p/100 = a/b$, rather than $a/b = p/100$, which was the standard way shown by the teacher and the textbook. This exchange revealed that one student understood the formula (understood that the two sides of an equal sign can be exchanged without changing the

[3] Japanese lesson 3 from the TIMSS Video Study (www.rbs.org/international/timss/resource_guide/lessons/by_country.php#japan) was an introduction to inequalities. After the teacher summarized the student solutions, he then presented a second, easier problem, which would allow all students to solve it using inequalities.

[4] From this behavior, it appeared some students did not understand the benefits of working with a partner (Gould, 2007), and there might not have been whole class discussion of the purpose of working collaboratively.

relationship), while the other student simply remembered the formula (Stacey, 2007, p. 45).

According to Gould (March, 2007), “Learning to argue about mathematical ideas is fundamental to understanding mathematics.” To be prepared to argue, students need to be able to listen to and respond to each other’s explanation of their work and thinking. The above issue was resolved, because the first student was able to listen to and understand her partner’s point of dissension.

Mathematical thinking

Although the term ‘mathematical thinking’ is used over 100 times in the *Principles and Standards of School Mathematics* (NCTM, 2000), no clear and explicit definition is provided. Stacey (March, 2007, pp. 39-40) described mathematical thinking as a “highly complex activity”, a process “…best discussed through examples.” Katagiri (March, 2007) also does not provide a clear definition, but he illustrates the logical steps (in order of complexity) of mathematical thinking for a counting problem that he used as an example (p. 115):

- Clarification of the meaning of the problem
- Coming up with a convenient counting method
- Sorting and counting
- Coming up with a method for simply and clearly expressing how the objects are sorted
- Encoding
- Replacing with easy-to-count things in a relationship of functional equivalence
- Expressing the counting methods as a formula
- Reading the formula
- Generalizing

Mathematical thinking is “the most important ability that arithmetic and mathematics courses need to cultivate in order to instill in students this ability to think and make judgments independently (p. 108)…“To be able to independently solve problems and expand upon problems and solving methods, the ability to use “mathematical thinking” is even more important than knowledge and skill, because it enables to drive the necessary knowledge and skill (p. 110). A working group composed of computer scientists and mathematicians offers a very general definition of mathematical thinking as “applying mathematical techniques, concepts and processes, either explicitly or implicitly, in the solution of problems.” (Henderson et al., 2001).

The ability to think and make judgments independently has been the goal of Japanese education since 1950, but it still remains to be achieved (Katagiri, p. 108). Such is the case also in the United States. As U.S. teachers turn to lesson study in mathematics to help them develop the ability to better understand and analyze student thinking and learning, they are finding they first need to understand how the students are thinking (or not thinking) about the mathematics they are being taught and then learn to move students from simply following and applying procedures in very rigid and limited ways to developing the ability to determine for themselves which procedures to use, how to achieve a level of efficiency in solving the problems, and whether what they have done makes sense.

Key window for considering mathematical thinking

The key window in this lesson study was communication, at the levels of teacher-to-teacher, teacher-to-student and student-to-student communications. In planning for the first teaching, there was no discussion of solution efficiency, and anticipation of student thinking and misunderstanding was limited. After observing the first teachings, the teachers discussed the need to probe more deeply students' understanding of the problem by offering counter-examples[5] to student solutions to push their thinking. For the second teaching it was agreed that students would be asked to consider if a 10% discount followed by a 20% discount was the same as or different from a 20% discount followed by a 10% discount. The students would also be urged to support their answer mathematically. After the two iterations of teaching, the teachers also began to focus on the need to help students consider how to solve problems by looking for student-generated efficient solutions and discussing them as a whole class.

At the level of student-to-student communication, the teachers began orchestrating more carefully the sharing of the student solutions, encouraging the students to communicate their solution strategies in a sequential fashion in order to enhance student understanding. The student presentations were planned to flow from the concrete to the abstract, from specific to general, from "ordinary solutions" to "efficient solutions." This teaching strategy was learned from watching a TIMSS video of a Japanese teacher orchestrating the student solution process (Hiebert, et al., 2003) prior to beginning the lesson study cycle.

What did teachers learn?

Subsequent to this first experience with lesson study, the teachers now report that in planning lessons, they think more carefully about anticipating students' solutions and orchestrating the manner in which the students communicate the solutions to the other members of the class. This is a change from the process previously in place, in which the teachers randomly selected students to come to the board to explain a solution to the problem. When the teachers used this practice (random selection versus planned selection

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[5] For calculating Ralph's discount, the teacher might ask why one couldn't first add 10% and 20% and then multiply the original price by 30%.

of student solutions), the flow of the lesson could be interrupted by "surprises" that could also confuse or misdirect students away from the learning objective.

Teachers reflected upon this first lesson study experience by responding to a series of questions (see Appendix 4). One of the main impacts of the lesson study cycle was to strengthen the teachers' ability to examine students working in the classroom and to discuss their observations, in turn making the teachers themselves more reflective thinkers, as documented in their questionnaire responses. Through the eyes of their colleagues, they learned more about their students' thinking; they obtained information about students beyond what was written on the student worksheets. Questions posed by colleagues during the post-lesson discussion caused them to rethink the approaches, activities and worksheets they used. Most importantly, the questions allowed them to consider the lesson and whether all that was planned and done really contributed to achieving the goals of the lesson.

Through practice made possible by all the teachers volunteering to teach the lesson, they became more proficient at observing lessons and collecting data on student thinking. Additionally, two teachers commented that due to their experience with lesson study they more carefully choose problems for both discussion and practice, look closely at the wording in selected problems to eliminate any ambiguity, and will better plan the sequence of problems on any future worksheets.

During the planning phase of the lesson study cycle, there was no detailed discussion of the scope and sequence (what students had learned in previous grades), accompanied by examination of the elementary textbooks and curriculum guide. However, it did highlight the teachers' previous strict adherence to the textbook, which in turn precipitated a subsequent review of the scope and sequence of the district's mathematics curriculum and the recognition of the need to align it with NCTM's *Focal Points* (NCTM, 2006). They recognized the need to use the "book more as a tool to help achieve the goal of the lesson and not to let the book become the goal." They also realized it was necessary to consider what students might have learned in previous years, how the concepts were taught, and what language was used in order to build upon students' prior knowledge and to understand the root of students' confusion.

Another realization was the need to move away from telling students too much to giving students an opportunity to come up with their own solution methods. To quote one teacher, "For true learning by the students, they need to be able to make or to see connections between what they already know and what it is we are trying to teach them." One teacher identified the lesson study process as an assessment tool that helps teachers see what students know about a topic and what knowledge they lack (misunderstanding).

Conclusions

Observers in classrooms often hear teachers ask students to “think.” Sometimes it is not clear about what and how students should be thinking. The APEC lesson study project, recognizing the intricacies in developing mathematical thinking, has devoted a series of conferences to the discussion of this very important topic. Observations of Japanese classrooms reveal the deliberate and explicit ways by which teachers help students learn and develop mathematical thinking skills; no steps are skipped, and no assumptions are made about student understanding.

Developing students’ mathematical thinking requires a coordinated group effort, as exemplified by the lesson study process. Teachers learn from colleagues’ data collected from observation of their students. The purpose of lesson study, however, is to inform daily instruction, when teachers are alone in the class with their students. By providing teachers with the opportunity to teach in front of colleagues and to collect data on student learning, thinking, and misunderstanding in colleagues’ classrooms, lesson study focuses teachers’ attention on how students interpret or misinterpret the lesson. Better understanding of students’ thinking can help teachers develop lessons that build students’ understanding rather than cause or contribute to their confusion.

Many teachers’ goal is to develop lessons that flow smoothly. However, a lesson that unfolds exactly as orchestrated may not shed light on real student thinking and understanding. The students in this lesson study cycle revealed to us a great deal about their misunderstandings and tenuous grasp of concepts, providing us with crucial information on the necessary next steps to correct their misunderstanding and to provide the scaffolding needed to build their understanding.

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Appendix 1: Ideal Students vs. Real Students

Ideal Students

* are prompt, polite, and prepared
* are respectful of each other and teacher
* persevere
* are motivated, interested, engaged
* are self-starters
* are self-reflective; engage in meta-cognition
* are active members of classroom discussions
* are responsible for their own learning
* take pride in their work
* are honest, have integrity

Real Students

* are unprepared: no tools, homework, mentally
* lack perseverance
* are unable/unwilling to think through problems
* are impulsive; act without thinking
* have a “tell me how to do it” attitude; just want to get it done
* don’t take time to assess reasonableness of answer
* exhibit varying levels of interest and perseverance within each classroom
* display lack of understanding
* have no concern for quality work
* are bored?
* are too motivated by grades
* don’t show thinking in writing (due to laziness?)

Were these lists written by the teachers in an effort to vent? Do they have real steps for turning their real students into ideal students?

Appendix 2:

Lesson Study at William Annin Middle School

Lesson study team members:
Patricia Gambino , Tara Gialanella, Chad Griffiths,
Mary Henry, Marian Palumbo, Elizabeth Slack

1. **Title of lesson: Assessing Student Understanding of Percent Concepts**

2. **Lesson Goals:** Students in grade 7 Algebra I from William Annin Middle School will:

   a. Understand the value for using efficient methods when solving percent problems
   b. Compare and contrast the relationship(s) between determining the "part" and "determining the "whole" in a percent problem

Class organization: Students will work with a partner to solve the problem. One student from selected pairs will put the solution on the board.

Rationale

Initially, as we worked with our seventh-grade students, we all became aware that our students did not have a deep understanding of the concept of percent. Moreover, it was clear that many students did not see the connection between fractions, decimals, percents, and proportions. Therefore, we decided to reexamine this concept. We felt that we needed to assess our students' current grasp of the topic of percent and uncover the sources of their misunderstandings and why they are not making the connections. It was at this point that it became clear that this topic, not the one we had originally chosen, should be the focus of our lesson study. Therefore, we decided that we would present our students with three problems involving percents and sale prices. Our students would have to decide at which store to buy an iPod in order to pay the lowest price. We chose this scenario, because we thought that it would grab our students' interests and be familiar to them. In addition, knowledge about and facility with percents is an important life-long skill.

Initially we presented the lesson to our regular seventh-grade mathematics students, some of whom did not find the lesson particularly challenging, as they were applying the same rote procedures they had learned in earlier grades. When we revised the lesson for the seventh-grade students enrolled in Algebra I, we realized we needed to give them additional opportunities to compare and contrast the various types of percent problems and to focus their attention on using efficient methods for solving the problems.

Scope and Sequence for Fractions, Decimals, Percents

Grade 2	Grade 3	Grade 4	Grade 5	Grade 6
Fraction equivalences (informal exploration)	• Fraction equivalences continued • Decimal concept introduced	• Operations with fractions/decimals introduced • Comparing, ordering decimals • Adding/subtracting decimals • Fraction concepts • Adding/subtracting fractions • Percents introduced • Convert fractions to decimals and percents w/ calculator • Multiply/divide decimals	• Add, subtract fractions • Multiply fractions using area model • Relate fractions, decimals, percents • Convert fractions to decimals, percents • Find percent of a number • Use unit fractions to find the whole • Use percents to interpret/create circle graphs	• Convert between fraction, decimal, percent • Review finding percent of a number • Use proportions to solve percent problems • Application: calculate tip, discounts, and sales tax

The Problem: The Carters are buying a new iPod Nano. Three stores have on sale this week the model they want, but they have decided to shop at Ralph's because they think Ralph's is offering a "double discount." Here are the ads. Did the Carters make a wise decision? Explain.

Radio Shop Original Price $172 Discount: ¼ off	Discount City Original Price: $180 Discount: 30% off	Ralph's Original Price $180 Discount: 10% off with an additional 20% off the discounted price

The management at Ralph's decided to change their ad to attract more customers. Here is the new ad:
Original Price $180. Discount 20% off with an additional 10% off the discounted price. How does this change the sale price? Explain.

3. Lesson Plan

Time	**Teacher Activity**	**Anticipated Student Thinking and Activity**	**Point To Notice and Evaluate**
0-3 min	Set up the problem and check for student understanding • Teacher discusses each store one at a time, displaying props • Clarify any student misunderstanding or questions		
3-8 min.	Tell students they should work on the problem with their partner. Both students are responsible for showing the solution strategies on their individual sheets of paper. When both students are finished they can begin work on the additional problem (different colored sheet of paper). • Teacher circulates to identify various solution methods for Ralph's only and	Ralph's .1 (180) = 18, 180 – 18=162 162 (.2) = 32.4, 162-32.4 = 129.60 order unimportant? .9 * .8 * 180 = 129.60 .72 (180) = 129.60	What distribution of students used different methods?

Time	Teacher Activity	Anticipated Student Thinking and Activity	Point To Notice and Evaluate
	selects students to record work on board. • While circulating, distribute second set of problems to be completed when students finish initial problem, (distribution method is optional)		
8-11 min	Poll students by show of hands How many think the Carters made a wise decision choosing Ralph's? How many think the Carters made a poor decision? Why might the Carters think that Ralph's would have the lower sale price? • Facilitate a class discussion	The ad is misleading The words say a 10% discount followed by a 20% discount, which means that first you have to multiply by 10%, find the sale price and then find the 20% discount from the sale price. A 10% discount followed by a 20% discount is not the same as a 30% discount – that is what the Carters were thinking.	What comments do students make about the Carters decision? What distribution of students thought the Carters made a wise decision? What distribution of students thought the Carters made a poor decision?
11-20 min	Students present their solutions for Ralph's • Teacher calls attention and facilitates a short discussion about the more "efficient solutions" • If necessary introduce solution .9(.8)(180) and .72(180)	.1 (180) = 18, 180 – 18=162 162 (.2) = 32.4, 162-32.4 = 129.60 .9 * .8 * 180 = 129.60 .72 (180) = 129.60	What distribution of students used efficient methods? What distribution of students
20-22 min.	Introduce the new ad – (show) The management at Ralph's	It doesn't change Multiplication is commutative	What distribution of students demonstrates

Time	**Teacher Activity**	**Anticipated Student Thinking and Activity**	**Point To Notice and Evaluate**
	decided to change their ad to attract more customers. Here is the new ad: Original Price $180, Discount 20% off with an additional 10% off the discounted price. How does this change the sale price? Explain • Facilitate a short discussion	.9 (.8) (180) = .8(.9)(180)	application of the commutative property to this example?
22-30 min.	Call the students' attention to the additional problem and have them continue to work on that one (computer problem): A computer is discounted 20% from its original price because it didn't sell. The store took an additional 30% off the discounted price. Barbara purchased the computer for $896. What was the original price of the computer? • Teacher circulates to collect solutions	896/.7/.8 = 1600 896 /.56 = 1600	What distribution of students used an efficient method? What distribution of students was able to transfer their knowledge of the first problem to this problem?
30-36 min	Teacher facilitates a discussion about the various solution methods and then summarizes by comparing and contrasting both problems, and generalizing .9 (.8) (180) = x .9 (.8) (whole) = part .8(.7) (x) = 896 .8 (.7) (whole) = part		

Time	Teacher Activity	Anticipated Student Thinking and Activity	Point To Notice and Evaluate
36-40 min	Teacher closes the lesson, asking the students to reflect on their learning and then complete the questions on the reflection sheet,		

Name ________________________________ Date__________________

The Problem: The Carters are buying a new iPod Nano. Three stores have on sale this week the model they want, but they have decided to shop at Ralph's because they think Ralph's is offering a "double discount." Here are the ads. Did the Carters make a wise decision? Explain.

Radio Shop Original Price $172 Discount: ¼ off	Discount City Original Price: $180 Discount: 30% off	Ralph's Original Price $180 Discount: 10% off with an additional 20% off the discounted price

Name:______________________ Date:_____

Try This One!!!

1. A computer is discounted 20% from its original price because it didn't sell. The store took an additional 30% off the discounted price. Barbara purchased the computer for $896. What was the original price of the computer?

Name ___________________ Date ______________

What mathematics did you learn or think about today?

In what ways was the lesson challenging?

In what ways was the lesson interesting?

Appendix 3: Template for data collection

Collection of Student Thinking

	Radio Shop Original Price \$172 Discount: ¼ off	Discount City Original Price: \$180 Discount: 30% off	Ralph's Original Price \$180 Discount: 10% off with an additional 20% off the discounted price
Use multiplication with decimal to find the discount and then subtract			
Use multiplication with fraction to find the discount and then subtract			
Use multiplication with (100-x)%			
Use multiplication with (whole-part) as a fraction			
Use a proportion to solve the problem with x/100			
Use a proportion to solve the problem with (100-x)/100			
Non-solutions for Ralph's – addition of percents			
Other solutions (note solution)			

Appendix 4: Assessing teacher learning during lesson study

1. Reflect on 1-3 things you learned from the lesson study experience.

2. What did you do prior to lesson study that hampered student learning?

3. What changes might you make to enhance student learning?

4. In what ways have you deepened your own understanding of mathematics?

5. What did you learn from observing colleagues' classrooms?

6. What did you learn from your colleagues' observation of your students?

7. What has changed since the lesson study cycle?

A LESSON THAT MAY DEVELOP MATHEMATICAL THINKING OF PRIMARY STUDENTS IN VIETNAM

FIND TWO NUMBERS THAT THEIR SUM AND A RESTRICTED CONDITION ARE KNOWN

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In Vietnam, after launching the national standard mathematics curriculum in 2006, the classroom mathematics teachers have learnt more on the innovative teaching strategies to implement more effective lessons focusing on mathematical thinking. The aim of this paper is to examine a lesson that we considered may develop mathematical thinking of primary students in Vietnam. A case study will be analysed using the observed students' activities in a videotaped lesson.

INTRODUCTION

In Vietnam, teachers encourage their students to invent their own procedures or algorithms for solving problems. The teachers use the teaching strategies that aim to:

- Promote active, initiative and self-conscious learning of the learners;
- Form and develop the ability of self-study;
- Cultivate the characteristics of flexible, independent, and creative thinking;
- Develop and practice the logical thinking;
- Apply problem solving approaches;
- Apply mathematics to real life situations.

In the teachers' guidebook for primary mathematics teachers at each grade there are four main activities in a lesson that teachers should follow to develop mathematical thinking:

Activity 1. Teacher manages students to work and achieve the following aims:

- Examine the students previous knowledge;
- Consolidate the previous knowledge involved with new lesson;
- Introduction to the new lesson.

Activity 2. Teacher facilitates students explore mathematical knowledge and construct new knowledge by themselves.

Activity 3. Students practice the new knowledge by solving exercises and problems in the textbook.

Activity 4. Teacher concludes what students have learnt from new lesson and assigns the homework.

Engaging to the lesson, the pupils will have opportunities to show their mathematical thinking through:

- The ability of observing, predicting, rational reasoning and logical reasoning;
- Knowing how to express procedures, properties by language at specific levels of generalization (by words, word formulas);

- Knowing how to investigate facts, situations, relationships in the process of learning and practicing mathematics;
- Developing ability on analyzing, synthesis, generalization, specifying; and starting to think critically and creatively.

In our point of view, the key windows for considering mathematical thinking are as follows:

- Students learn mathematical concepts with *meaningful understanding*;
- Students construct *individual algorithm* and *techniques* themselves with understanding to solve some specific problems;
- Students use learnt mathematics to *solve* mathematical problems effectively;
- Students show mathematical thinking by *communicating* (talking, writing, arguing, discussing, and representing);
- Students *reflect critically* their mathematical thinking in order to improve their learning;
- Mathematical thinking is *social* and *relative* to each individual student;
- Students apply *logical* and *systematic* thinking in mathematical and other contexts;
- Students use *thinking operations* in solving problems: *comparison*, *analogy*, *generalization*, and *specialization*;

The lesson which will be analyzed in this paper is prepared by classroom teacher for grade five primary students. We can find from the lesson plan the three main tasks and a quiz proposed in the lesson:

Introductory Task. Use 2 cm-cards and 4 cm-cards to make a toy train of 5 wagons?

Task 1. Use 2 cm-cards and 4 cm-cards to make a train with the length of 16 cm?

Task 2. A train with the length of 50 cm including 20 wagons, how many red wagons and blue wagons are there?

Task 3. A train with the length of 100 cm including 36 wagons, how many red wagons and blue wagons are there?

Quiz (Homework). There are 33 liters of fish sauce contained in 2 liter-bottles and 5 liter-bottles. The number of bottles used is 12. Find the number of 2 liter-bottles and 5 liter-bottles used. Given that, all bottles are full of fish sauce.

At the end of Grade 4, students know how to solve and express solutions of problems having three operations of natural numbers.

Example. A toy train has 3 wagons with the length of 2 cm, and 2 wagons with the length of 4 cm. Find the length of the train?

Answer. $3 \times 2 + 2 \times 4 = 14$ (cm).

But in the second semester of grade 5, if we set the problem in a reverse way:

A toy train has two types of wagon: 2 cm-wagons and 4 cm-wagons. This train has the length of 14 cm including 5 wagons. Find the numbers of 2 cm-wagons and 4 cm-wagons of the train.

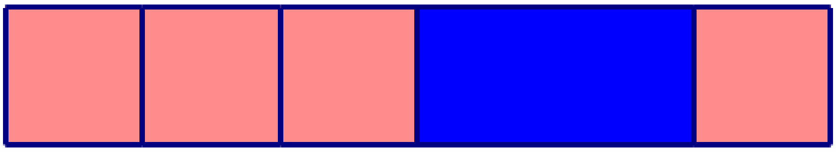
The sum of two numbers needed to find is 5.

Restricted condition: The total length of 2 cm-wagons and 4 cm-wagons is 14 cm.

This reverse problem is quite different with what students have learnt in grade 4. The problem requires them to analyze a natural number into sum of two other numbers satisfying a restricted condition logically.

ANALYSIS OF THE TASKS

Analysis of Introductory Task. Use 2 cm-cards and 4 cm-cards to make a toy train of 5 wagons?



This task is an introductory activity. It is an open-ended task that requires pupils to make many trains as possible. Pupils can arrange the cards to make a train, use the strategy "*guess and check*" to get many answers. To solve this task mathematically teacher guides students to make a systematic list of all abilities.

N. of red wagons	0	1	2	3	4	5
N. of red wagons	5	4	3	2	1	0
The length of the train in cm	20	18	16	14	12	10

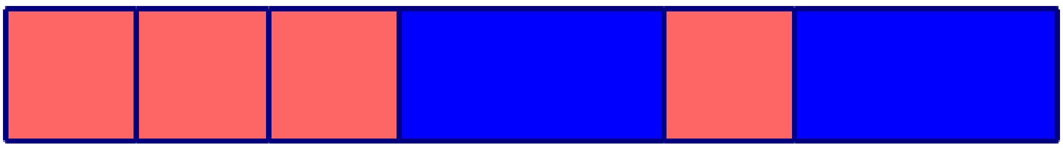
From the above table, students recognize the relationship between the length and the numbers of red wagons, blue wagons. If the number of red wagons increases one, then the length of the train decreases 2 cm. In this task, students know that:

N. of red wagons + N. of blue wagons = 5

There are 6 options for this task. If the length of the train is given then we can find exactly the N. of red wagons and N. of blue wagons. The length of the train is understood as a restricted condition. Students will see that the train has the longest length 20 cm when all of the wagons are blue and shortest length 10 cm when all of the wagons are red.

The aim of this introductory task is to help students recognize the restricted condition in finding two numbers that their sum is known.

Analysis of Task 1. Make a train with the length of 16 cm.



This is also an open-ended task that requires students to make a systematic list of all abilities. The restricted condition is given but the sum of two numbers is unknown.

N. of red wagons	8	6	4	2	0
N. of blue wagons	0	1	2	3	4

Total of wagons	8	7	6	5	4

There are 5 answers to this task. Students know how to analyze a natural number into sum of two natural numbers with a specific restricted condition.

$16 = 8 \times 2 + 0 \times 4$ $\quad$ $16 = 4 \times 2 + 2 \times 4$ $\quad$ $16 = 0 \times 2 + 4 \times 4$

$16 = 6 \times 2 + 1 \times 4$ $\quad$ $16 = 2 \times 2 + 3 \times 4$

From the table students will see that a train with the length of 16 cm including 6 wagons has 4 red wagons and 2 blue wagons. The restricted condition of this problem is:

Sum: N. of red wagons + N. of blue wagons = 6.

Restricted condition: The length of the train is 16 cm.

If all wagons are red then the length of the train decreases: $16 - 6 \times 2 = 4$, then the number of blue wagons: $(16 - 6 \times 2) \div 2 = 2$.

Students practice this procedure to consolidate what they have learnt. The most important fact that the students need to realize is the difference 2 cm between one blue wagon and one red wagon.

Analysis of Task 2. A train with the length of 50 cm including 20 wagons, how many red wagons and blue wagons are there?

In this task, teacher does not ask students to make a table but encourages them to generalise what they have observed in some concrete situations above to create their own procedure to solve the general problem.

Students make temporary assumption: If the train has only red wagons, the length of the train decreases:

$$50 - 20 \times 2 = 10$$

The number of blue wagons: $(50 - 20 \times 2) \div 2 = 5$.

Students look back the solution by checking their answer: $15 \times 2 + 5 \times 4 = 50$ cm.

Analysis of Consolidation Task. A train with the length of 100 cm including 36 wagons, how many red wagons and blue wagons are there?

The aim of this task is to help students consolidate what they have studied. They use their procedure to solve this problem by using temporary assumption.

The number of blue wagons: $(100 - 36 \times 2) \div 2 = 14$.

Analysis of Quiz. There are 33 liters of fish sauce contained in 2-litter bottles and 5-liter bottles. The number of bottles used is 12. Find the number of 2-liter bottles and 5-liter bottles used. Given that, all of bottles are full of fish sauce.

This is an application task. Students can solve this task as homework. Students learn how to apply what they have studied from the lesson to solve a realistic problem. Students recognise that the difference between one 5-liter bottle and one 2-litter bottle is 3 liters.

The number of 5-liter bottles: $(33 - 12\times2) \div 3 = 3$. Thus, the answer is 9 two-liter bottles and 3 five-liter bottles.

ANALYSIS OF THE VIDEOTAPED LESSON

The lesson is videotaped and analyzed using the video recording and the transcript. The actual lesson included several activities. The analysis in this section will be conducted by dividing the actual lesson into three stages: introductory activities, activities for task 1, and activities for task 2 and task 3. Each stage will be described and analyzed.

Introductory activities

1. Students were asked to make a train of 5 wagons by using 2 cm-red cards and 4 cm-blue cards.
2. Students were asked to use only 5 cards to make their own trains.
3. Students discussed in small groups of 4 students to list as many abilities as possible.
4. Students had to recognize the relationship between the length of the train and the numbers of red wagons, blue wagons.
5. Students had to recognize the restricted condition for each specific case in finding two numbers that their sum is known.
6. From the established table students had to understand that: if the number of red wagons increases one, then the length of the train decreases 2 cm.

In the lesson, some students made only one train of 5 wagons as required and then stop working.

Teacher asked students to paste their answer on blackboard. Most of the answers were presented except the two last options: 5 red wagons and 0 blue wagon, or 0 red wagon and 5 blue wagons

Teacher asked students to arrange the data following a systematic list.

S: There are many answers to this task.

T: Can you check your answer?

S: My train has 5 wagons including 2 red and 3 blue wagons. The length of the train is: $2 \times 2 + 3 \times 4 = 16$ cm.

T: We call "*the length of the train*" the restricted condition. Can you identify another restricted condition?

S: 14 cm.

T: How many red wagons and blue wagons in this train?

S: 3 and 2. We have $3 \times 2 + 2 \times 4 = 14$ cm.

Activities for Task 1

1. Students were asked to make a train with the length of 16 cm by using 2 cm-red cards and 4 cm-blue cards.
2. Students were asked to use some cards to make their own trains with the same length of 16 cm.

3. Students discussed in small group of 4 students to list as many abilities as possible.
4. Students had to recognize the relationship between the fixed length of the train and the numbers of red wagons, blue wagons.
5. Students had to know to analyse a natural number into sum of two natural numbers with specific restricted conditions.

In this task, students received an almost blank table; some cells have numbers that helped students fill the data into the table easier.

Số toa ĐỎ	8	6	4	2	0
Số toa XANH	0	1	2	3	4
Các toa: TỔNG	8	7	6	5	4

Students analysed number 16 as follows:

$16 = 8 \times 2 + 0 \times 4$

$16 = 6 \times 2 + 1 \times 4$

$16 = 4 \times 2 + 2 \times 4$

$16 = 2 \times 2 + 3 \times 4$

$16 = 0 \times 2 + 4 \times 4$

T: What is given?

S: The length of the train is 16 cm.

T: If the train has 6 wagons, how many red wagons and blue wagons in this train?

S: From the table I saw that this train has 4 red wagons and 2 blue wagons.

T: If we do not make the table, can you explain your solution?

S: If all 6 wagons are red, the train's length decreases 4 cm. So I got 2 blue wagons.

T: Who can express the answer by using mathematical operations?

S: $(16 - 6 \times 2) \div 2 = 4 \div 2 = 2$ (blue wagons).

Activities for Task 2 and Task 3

1. Students were asked to solve an extended problem that is difficult to guess and check.
2. Students were required to create a procedure to solve the task with a specific restricted condition.
3. Students were asked to present their answer by using mathematical operations?

In this task, some students used mental calculations or "*guess and check*" strategy to find out the answers. But they could not explain the answer logically.

S: There are 15 red wagons: $15 \times 2 = 30$ cm. And 5 blue wagons: $5 \times 4 = 20$ cm.

T: I ask you to give a procedure to solve this task not only use your mental calculation.

The teacher guided students to create a procedure by using the temporary assumption to solve the problem.

T: If 20 wagons are red, what is the length of the train?

S: 40 cm.

T: Why does the length decrease?

S: Because we replaced blue wagons by red wagons?

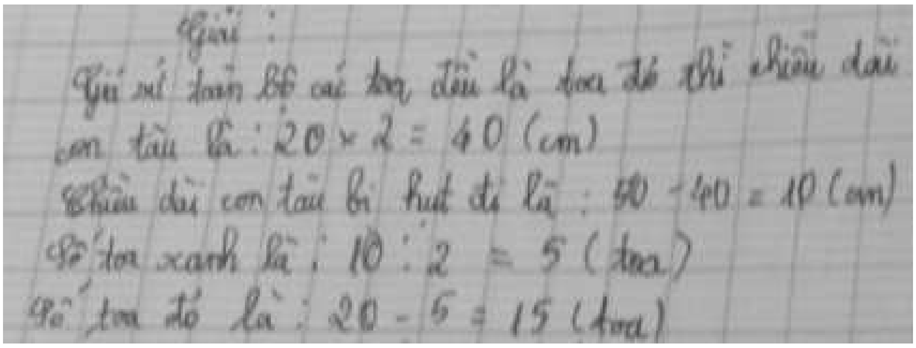
T: How many blue wagons did we replace?

S: 5 blue wagons.

T: How did you get 5?

S: (50 - 40) ÷ 2 = 5.

A student's solution:



Giải:

Giả sử toàn bộ cả toa đều là toa đỏ thì chiều dài con tàu là: 20 × 2 = 40 (cm)

Chiều dài con tàu bị hụt đi là: 50 - 40 = 10 (cm)

Số toa xanh là: 10 : 2 = 5 (toa)

Số toa đỏ là: 20 - 5 = 15 (toa)

Translation into English of a student's solution:

If all wagons are red then the train's length is: 20 × 2 = 40 (cm).

The train's length decreases: 50 - 40 = 10 (cm).

The number of blue wagons: 10 ÷ 2 = 5 (wagons).

The number of red wagons: 20 - 5 = 15 (wagons).

The students applied the procedure to solve Task 3.

The number of blue wagons: (100 - 36 × 2) ÷ 2 = 14 (wagons).

The number of red wagons: 36 - 14 = 22 (wagons).

DISCUSSION AND CONCLUSION

Teaching primary school mathematics aims to equip young pupils with basic mathematics skills and develop their mathematical thinking to solve problems. Some senior classroom teachers have experienced to foster and develop students' mathematical thinking without theoretical background. Most of teachers in Vietnam really need a practical framework to develop pupils' mathematical thinking in their actual classrooms.

This lesson was prepared by a senior teacher, he has involved in some educational projects at primary level. As I analyzed the activities in the lesson by using videotaped recording, the teacher followed four main activities in a lesson that were suggested by the MoET to develop mathematical thinking.

In introductory activities, the task is a open-ended task, it helped students get start to observe many abilities, predict the length of the train by "guess and check". Teacher managed students to work and achieved the following aims:

- Examine the students' previous knowledge in finding the answer for:

  □ × 2 + ○ × 4 = ?

  where □ + ○ = 5.
- Consolidate the previous knowledge involved with new lesson: Find two numbers that their sum is 5. The new lesson needs to have one more restricted condition is the length of the train.

Find □ and ○ such that: □ × 2 + ○× 4 = 14 and □ + ○ = 5.

These activities gave students opportunities to show the ability of observing, predicting, rational reasoning and logical reasoning in solving problems related to the analysis of a natural number into the sum of two other numbers with a restricted condition.

In activities for task 1, teacher facilitated students explore mathematical knowledge and construct new knowledge by themselves. Students recognized the relationship between the fixed length of the train and the numbers of red wagons, blue wagons.

Find □ and ○, where □ × 2 + ○× 4 = 16, but □ + ○ = unknown. Students observed and predicted answers. Students created a procedure to solve the problem when □ + ○ = a fixed number. From specific situations students suggested a procedure to solve general problem. Students invented their own procedures or algorithms for solving problems.

In activities for task 2 and task 3, students practiced the new knowledge by solving exercises and problems given by teacher.

Students applied the analysis of natural number into sum of two other numbers with a restricted condition to solve some mathematics problems systematically by using temporary assumption. These two tasks examined the thinking operations that occurred in the lesson such as: comparison, generalization, and specialization.

Teacher concluded what students have learnt from new lesson and assigned the homework.

Acknowledgement. This lesson study was conducted in Hue City, Vietnam under the collaborative framework involving mathematics education among the APEC Member Economies. Special thanks due to the principal, mathematics teachers of the primary school Le Qui Don, Hue City, Vietnam for their contribution to the research.

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Appendix

Mathematics Lesson Plan

Grade 5 (10-11 years old)

Teacher: Senior Teacher Mr. Tran Quang Khen, Le Qui Don Primary School, Hue City, Vietnam.

1. **Title**: Find two numbers that their sum and a restricted condition between them are known.
2. **About the research theme**
   - Nurturing ability of observing, predicting, rational reasoning and logical reasoning in solving problems related to the analysis of a natural number into the sum of two other numbers with a restricted condition.
   - Examining instruction that focuses on "applying the analysis of natural number into sum of two other numbers with a restricted condition to solve some mathematics problems systematically by using *temporary assumption*".
   - Examining the thinking operations that occur in the lesson such as: comparison, generalization and specialization.

In the national standard mathematics curriculum (2006) for primary level, we emphasize more in word problems that considered being good situations for pupils to explore and solve mathematical problems. Students' mathematical thinking will be enhanced when they solve word problems. Most of these problems are rooted from the real life situations.

At the beginning of Grade 4, students know how to solve and express solutions of problems having three operations of natural numbers.

Example. A toy train has 3 wagons with the length of 2 cm, and 2 wagons with the length of 4 cm. Find the length of the train.

Answer. $3 \times 2 + 2 \times 4 = 14$ (cm).

But if we set the problem in a reverse way:

A toy train has two types of wagon: 2 cm- wagons and 4 cm - wagons. This train has the length of 14 cm including 5 wagons. Find the numbers of 2 cm- wagons and 4 cm - wagons of the train.

The sum of two numbers needed to find is 5.

Restricted condition: The total length of 2 cm-wagons and 4 cm-wagons is 14 cm.

This reverse problem is quite different with what students have learnt before. The problem requires them to analyze a natural number into sum of two other numbers logically.

3. **Goal**
   - For students to be able to recognize a number as a sum of two other number with a restricted condition;
   - Know how to find two numbers that their sum and a restricted condition are known.

4. Instruction plan

- Understanding the relationship of two related quantities;
- Identifying the restricted condition of the relationship of two quantities.

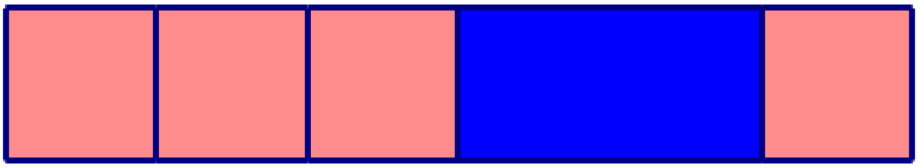
5. Instruction of the lesson

(1) Goal

- For students to realize that making a systematic list will help them understand the problem intuitively;
- Look for a pattern or procedure to solve a set of mathematical problems by using temporary assumption;
- Generalize the procedures obtained to solve some realistic problems;

(2) Flow of the lesson

Teacher prepares some red cards of 2 cm × 2 cm, and some blue card of 4 cm × 2 cm. Teacher called these cards to be wagons of a toy train.



Students are to use cards to make their own train.

Instructional Activities	Points for Consideration
Introductory Task. Use 2 cm cards and 4 cm cards to make a toy train of 5 wagons? List all abilities. How many answers can we get for this problem?	This is an open-ended task that requires students to make a systematic list of all abilities. Guess and check to get many answers.

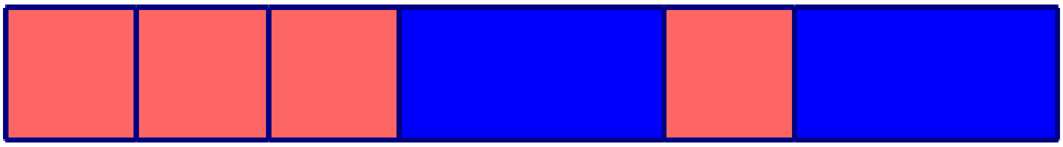
Students fill the data into the following table.

N. of red wagons		1		3		
N. of blue wagons	5		3	2		0
The length of the train in cm	20					

From the table of data, teacher supports students with following guided questions.

Question 0.1: If the number of red wagons increases one, then what is about the length of the train?	Recognize the relationship between the length and the numbers of red wagons, blue wagons. Recognize the difference between blue wagon and red wagon is 2 cm.
Question 0.2. A train with the length of 14 cm including 5 wagons. How many red wagons and blue wagons are there?	Understand with a restricted condition the answer will be unique.
Question 0.3. When the length of the train is	Identify the restricted condition for

longest? Shortest?	each case.
Task 1. Make a train with the length of 16 cm. List all the abilities and find the number of wagons in your trains?	Know how to analyze a natural number into sum of two natural numbers. Recognize the number of red wagons is an even number. Express the relationship between two quantities: If the number of blue wagons increases one the number of red wagons decreases two. This is also an open-ended task that requires students to make a systematic list of all abilities.



Students fill the data into the following table.

N. of red wagons	8			2	
N. of blue wagons		1			4
Total of wagons			6		

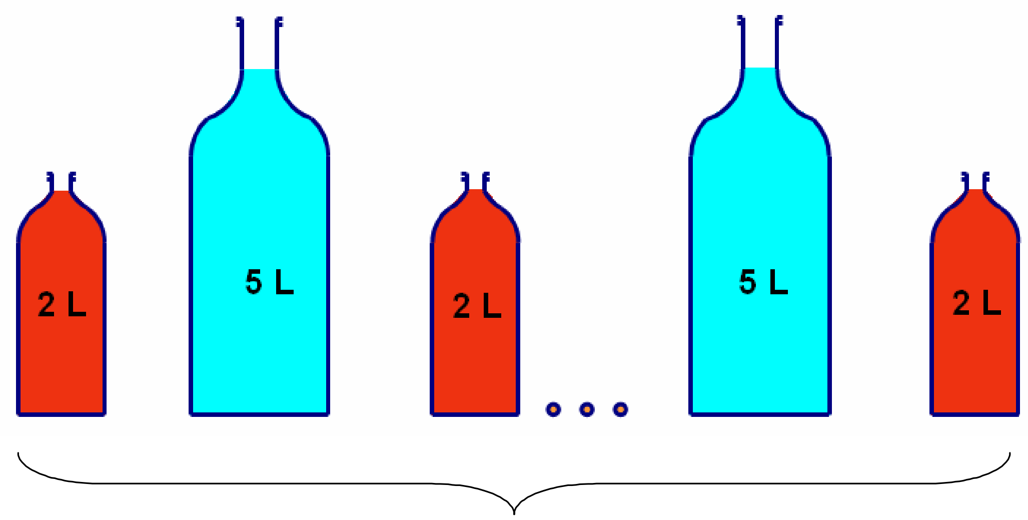
From the table of data, teacher supports students with following guided questions.

Question 1.1: A train with the length of 16 cm including 6 wagons. How many red wagons and blue wagons are there?	Predict the pattern to solve the general problems.
Question 1.2: When does the train have largest number of wagons? Smallest number of wagons?	Identify the restricted condition for each case.

Teacher encourage students create their own procedure to solve the problem.

Task 2. A train with the length of 50 cm including 20 wagons. How many red wagons and blue wagons are there?	The number of blue wagons: $(50 - 20\times2) \div 2 = 5$ Thus, the answer is 15 red and 5 blue wagons.
Question 2.1. If the train has only red wagons, what is the shortened length of the train? *Question* 2.2. In this case, you do not make a table. How can you find the number of blue wagons?	Create a procedure to solve the general problem.
Task 3 (Consolidation Task). A train with the length of 100 cm including 36 wagons, how	The number of blue wagons:

many red wagons and blue wagons are there?	(100 - 36×2) ÷ 2 = 14 Thus, the answer is 22 red and 14 blue wagons. Students practice the procedure just created to solve this problem.
Quiz. There are 33 liters of fish sauce contained in 2-liter bottles and 5-liter bottles. The number of bottles used is 12. Find the number of 2-liter bottles and 5-liter bottles used. Known that all bottles are full of fish sauce.	The number of 5-liter bottles: (33 - 12×2) ÷ 3 = 3. Thus, the answer is 9 two-liter bottles and 3 five-liter bottles. Students apply what they have learnt from the lesson to solve a realistic problem. Asking students to solve another problem to evaluate what the students are learning.



12 bottles containing 33 liters