DEVELOPING MATHEMATICAL THINKING THROUGH LESSON STUDY: INITIAL EFFORTS AND RESULTS

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This paper describes how through lesson study two teachers were made to experience mathematical thinking so that they in turn could create opportunities for their pupils to experience it. In developing the lesson intended for this, they realized the need to deviate from many of their long-held unquestioned practices. This meant trying out what they have not done before in their teaching. Despite some lapses due to their adjusting to the changes in their practices, their actual teaching of the lesson showed that their initial efforts could engage pupils in mathematical thinking. And so, this was a good start.

ANALYZING EXISTING CONDITIONS

Identifying the Usual Practices

The usual components of an elementary mathematics lesson are: drill, review, presentation, developmental activity, fixing skills, generalization, application, and evaluation. In the presentation, a word problem serves as a source of the numbers that are computed by the whole class through the guidance of the teacher during the developmental activity. After the pupils have read the word problem that is expressed in English, it is a standard procedure that they are made to answer guide questions to help them understand and analyse it. Filipino is the native language but English is used to teach mathematics. The guide questions are: (1) What is asked (A)? (2) What is given (G)? (3) What is the word clue/operation to use (O)? (4) What is the number sentence (N)? (6) What is the answer (A)? AGONA implicitly shows how a word problem should be analysed. (Department of Education, 2002).

Determining the Need for a Research Lesson

Last year, the teachers identified that their lesson study goal is to increase pupils' motivation in learning mathematics and to improve their comprehension and analysis of word problems. For this year, they wanted to develop a lesson on solving problems involving subtraction of whole numbers with regrouping because it is a very difficult topic for many pupils. Difficulty is on understanding a foreign language and also on the process of regrouping. To help pupils understand and analyse word problems, teachers code switch and allow the pupils to talk in Filipino or code switch. They teach pupils to look for clue words, words that are associated with a particular operation, so that they will know what operation to use. Examples are deduct, reduce,

less, and take away which are associated with subtraction. Relative to forming a number sentence involving subtraction, teachers observe that pupils tend to write the number mentioned first in the word problem as the minuend even if this is actually the subtrahend or the difference. Apparently, the pupils do not understand the word problem. Moreover, they note that in finding the difference of two whole numbers with several digits, there are pupils who always subtract a smaller digit from a bigger digit with the same place value disregarding whether the smaller digit is in the minuend or subtrahend. In effect, they do not use regrouping. The lesson study group has to develop a research lesson that takes into account all these conditions and at the same time develop mathematical thinking among pupils.

Examining the Usual Practices

Since grade 1, pupils have been made to do AGONA as a whole class and it takes time. The teachers were asked to consider other ways to help pupils understand and analyse word problems. These include making them relate what they understand about a it using their own words or asking them what the answer to it is and to explain how they got this. These require pupils to make sense of the word problem. The teachers were also challenged to make their pupils solve the word problem on their own without their guidance. They predicted that many would not be able to do it but a few would. But they were willing to try. They were reminded of the importance of enabling pupils to think on their own.

The teachers were also asked why they teach their pupils to rely on clue words to determine the operation to use in a word problem. It was explained to them that this might not really help pupils' comprehension for they might just look for the clue-words and no longer try to understand the word problem. They were given an example where there was no clue word. (There are 3 500 balls. There are 1 750 balls that are not in the boxes. How many balls are in the boxes?) They were also given a counterexample where the word that is thought to be associated with a particular operation is not so. (The grade 4 classes collected 2 478 stamps. The grade 3 classes collected 1 543 stamps. How many more stamps did the grade 4 classes collected than the grade 3 classes?). Lastly, they were made to realize that word clues are often limited only to the "take away" interpretation of subtraction and disregards its other meanings, namely: additive, comparative, partitive, and incremental (Troutman & Lichtenberg, 1991). Since the teachers were not aware of these other interpretations, they were asked to write word problems on them.

DEVELOPING A LESSON THAT ELICITS MATHEMATICAL THINKING

Using a Framework on Mathematical Thinking

In developing the lesson, the following were considered: teachers must engage in

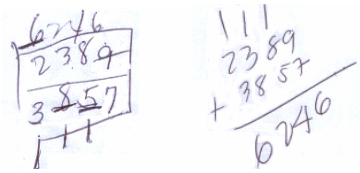
mathematical thinking so that they can elicit this also in pupils; teach mathematics through problem solving to integrate mathematical thinking in the learning of content and enhance pupils' reasoning and communicating skills; encourage pupils to think through questioning; ask pupils to discuss their ideas and formulate their own problems; use non-routine tasks such as open-ended problems to develop mathematical thinking in routine tasks; enable pupils to use their previous knowledge and skills to unfamiliar contexts; connect concepts and procedures; anticipate various pupils' responses; and develop mathematical attitudes like persistence in solving problems and verifying results (APEC Organizing Committee 2006) .

Engaging Teachers in Mathematical thinking

Considering that the procedure in subtracting whole numbers with regrouping had been introduced since grade 3 only with fewer digits, and assuming that pupils could understand English, then this topic would not present anything new from what had already been taken previously. And where would mathematical thinking naturally fit in? So the teachers were probed on what other lessons they had taught so far about subtraction. They had given exercises on finding any of the following, given the other two: difference, subtrahend, or minuend. Building on this, the teachers were then asked in which word problem the pupils would have more opportunity to think - finding one of the following given the other two: difference, subtrahend, or minuend or finding the missing digits in the minuend, subtrahend, and difference which are all given. They were then made to answer the following:

Mr. Jose saves money for his house repair. The repair costs P _246. He has already saved P238_. So he still needs to save P3_ _7. How much does the house repair cost? How much has Mr. Jose saved already? How much more does he need to save?

Shown below is the work of one teacher who consistently used addition to find the missing digits.



The other teacher used a combination of addition and subtraction. Both of them clearly explained their work. They used the relationship of addition and subtraction, the concept of place value, and the process of regrouping. They appreciated solving this word problem involving missing digits and their using different ways to find them. They remarked that this was the type of word problem that could make their pupils think more. Thus they were asked to make similar word problems using the different meanings of subtraction. It was intended that later they would be given an

example of an open-ended problem involving numbers with missing digits.

So the teachers made word problems. They were systematic in their explorations. One placed a blank in each column of the number sentence from right to left and in the subtrahend, then minuend, and the difference. The other placed a blank in each column of the number sentence except in the tens place where he placed a blank each in the minuend and the difference. The blanks were distributed in the minuend, subtrahend, and difference. To his surprise, he got many different possible answers. He asked that if this was the case, then what would the correct answer be. His question provided the opportunity to introduce to the teachers what open-ended problems are. He was told that the problem he made was one example. He said that in previous seminars that he had attended, he had already encountered this term. But it was only then that he understood what it meant. Since the teachers engaged in mathematical thinking, they were better prepared to engage their pupils in this experience, too.

While developing the lesson, the teachers realized that the pupils would need time to solve problems. So they decided not to have a drill and review and to have the problems right away. The problems that the teachers made are shown in the lesson plan. Together, the group thought of possible ways that the pupils might solve them.

Preparing the Task

The task consisted of two problems. Problem 1 was supposed to familiarize pupils with subtracting numbers with missing digits. Problems 1 and 2 were similar because their numbers both had missing digits, and they could be solved in different ways. They were different in that the number sentence in Problem 2 had a column with two missing digits and so was open-ended while in Problem 1 each column of the number sentence had only one missing digit and so was not open-ended. Problem 2 had many different correct answers while Problem 1 had only one. Problem 2 used the comparative meaning of subtraction while Problem 1 used the partitive meaning. In Problem 2 a smaller number came first before a bigger number while in Problem 1, a bigger number came first before a smaller number.

In developing the lesson, it was mentioned that a pattern can be observed as one systematically replaced the two blanks with different digits in the tens column of the number sentence in Problem 2. However, time was not enough to go deeply into this. In the lesson itself, it would be enough that pupils would realize that they could substitute different digits and get different correct answers. This would be the first time that they would be solving a problem like this. The teacher themselves would have to investigate the following: If one systematically substituted the digits 0 to 9 in the minuend, then the digits in the difference would be the same. Why? The higher the digit that was substituted in the minuend, the higher would be the digit in the

difference. Why? The higher the digit that was substituted in the difference, the higher would be the digit in the minuend. Why? Would these results still be true if the given digit was no longer 9? Why? Why was it that it was only in the tens place of the minuend and difference that the digits differed? Why were the digits the same for all the other place values? What would be the answers if instead of the digits missing on the minuend and the difference they were missing on the minuend and the subtrahend or the subtrahend and the difference? What if instead of two digits on the same column, two consecutive digits on the same row were missing? These questions show that the task is mathematically rich.

ELICITING MATHEMATICAL THINKING IN PUPILS

Problem 1

By asking a question that relates to their daily life experience, the teacher got the pupils' interest in the lesson. Being interested can facilitate thinking. By making the pupils read the situation, in particular the numbers that have missing digits, they were made to think because this was the first time that they had something like this. They needed to relate their understanding of place value in order to read correctly each number. Some of them were able to. This was an instance of connecting their existing knowledge to a new context. By making them write the numbers, he focused their attention to the numbers' having missing digits and tried to make them realize the need to find these digits first to answer the questions. Focusing is an important thinking skill.

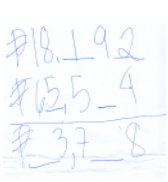
The pupils were not asked to answer AGONA. They were expected to understand what they read and figure out on their own how to answer the questions. Understanding the situation and analysing the relationships implied there without the teacher's guidance required thinking. This was not what they were used to do. Moreover, there were no clue words mentioned on which they could rely. When the teacher asked if they could form a number sentence for the situation, they at first said no. When he raised the question again, they hesitated to answer until he said that there was a problem expressed in what they read. This was the first time that they encountered a problem presented in this way. The number sentences that they had formed before were for those problems that they were familiar with. In what they had now, every number needed was already given but had missing digits. What they had done previously involved two numbers without missing digits and they had to find a third number by applying a given operation on them. So they must be thinking how they would connect what they knew with this new one. If they truly understood what they read, then they could write a correct number sentence that is, represent the relationship symbolically, which is an important thinking skill. In particular, as the video shows, the pupil who was called to write the number sentence was unsure of what he was doing. But with the teacher's coaching, he was assured that what he was

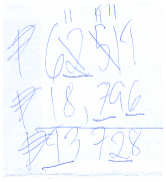
doing was right. This action of the teacher was important. He did not give the number sentence himself but gestured approval of the work of a doubting pupil.

After one correct number sentence was given, the teacher asked the class to work in pairs and discuss. Everybody worked and most of those who had seatmates discussed with each other. This discussion provided opportunity for the pupils to think together in determining the digit that should go into each blank. Then, he asked some pupils to explain their work. Explaining one's work required thinking. One had to be able to clearly, completely, and correctly describe one's work in order for others to understand and be convinced that the work was correct. Some explanations were more conceptual than the others. The pupils gave reasons for how they obtained the digits using the inverse relationship of addition and subtraction, the concept of place value, and the process of regrouping. It looked like this problem was easy for most students. Based on the numbers they obtained with completed digits, the teacher asked if they could give another number sentence. He also remarked that if they knew any two numbers, then they could get the third number. The pupils added the subtrahend and the difference and got the minuend. Indirectly, he made them verify if the numbers they got were correct. Implicitly, he also emphasized that addition and subtraction are inverse operations and when two numbers are known, a third number can be known given an operation. Later, the pupils would meet the latter relationship again in the next problem. He also established that now that the numbers did not have missing digits anymore, what they had done before with such numbers could be applied with these ones, too. Recognizing and making all these connections are important mathematical thinking skills. Forming the habit of always checking one's answers is very important. He commented that all their answers were correct although they had different solutions referring to the ways that they used to find the missing digits.

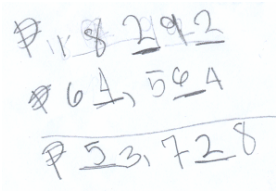
Problem 2

Basically, the teacher's approach was the same as that in the first problem except that he did not ask the pupils to write each number anymore. And so his actions that elicited thinking in them earlier did the same here because in this problem, the numbers were larger and a smaller number was mentioned first before a larger number. When asked to write a number sentence, a pupil wrote this smaller number first then the larger number. Perhaps, because he was not used to being left on his own to understand and analyse a word problem, he did not fully understand it. No one called the attention of the teacher about this nor questioned him or the pupil who wrote it why this was so. Only one pupil realized the error. As shown below, he wrote the bigger number first followed by the smaller number on his paper even before the teacher called their attention.

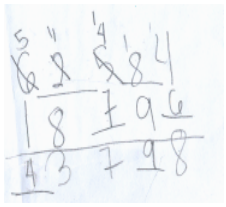
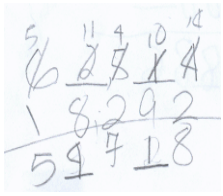
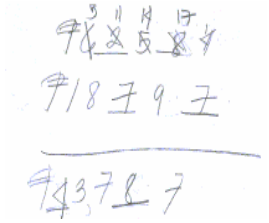
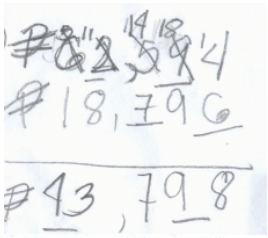
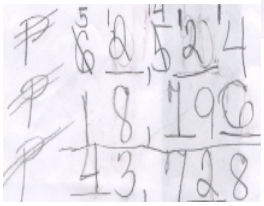
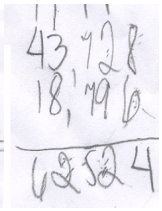
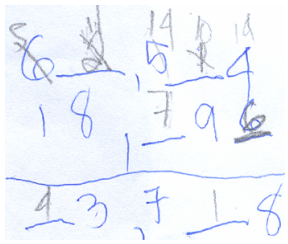
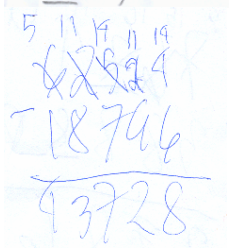




This incident provided an opportunity for pupils to really think. Every one tried to make sense of the number sentence and how to carry out subtraction to get the missing digits. According to an observer, a pupil adjusted the alignment of digits so that 6 is under 8 and so 18 was bigger than 6. Then she affixed 0 at the units-digit of the number above so as to align all the digits correspondingly with those in the number below. Others subtracted the digits in the number below from their corresponding digits in the number above from right to left. When they reached the ten thousands place, they reversed the direction from top to bottom as shown below.



When the teacher realized the mistake, he called the attention of the class by asking which number was bigger. He did not say that the number sentence was incorrect. He wanted the class itself to realize this. And the boy who had corrected the mistake earlier said that the second number should be on top because it was bigger and the number above should be the second number because it was smaller. But apparently, the teacher wanted a conceptual reasoning like this: All the numbers have the same number of digits and the second number has the highest ten thousands digit so it should be written first for the subtraction sentence to be correct. So his question was intended to make pupils think why the number sentence was incorrect. They simply responded “6.” A girl wrote a different number sentence. She must have reasoned that the largest number should be the sum of the two smaller numbers so a number sentence could be one that involved addition. However, he was trying to correct the subtraction sentence, so he did not pursue her answer. Nevertheless, this shows that pupils had different number sentences in mind and that they understood relationships. This indicated that they were thinking. Finally, the number sentence that he expected was written on the board. And the pupils again worked on their seats. Shown below are the works of several pupils. Although some of their answers were incorrect, still these showed that they had an idea that they could substitute different digits to the blanks in the tens place of the minuend or difference. Most of these pupils checked their answers using addition.

Then, the pupils were asked to show and explain their work. It was observed that a pair of pupils was not following the class discussion because they were so preoccupied with finding the missing digits and checking their work. It was not clear in the explanations of those who were called how they determined the missing digit in the tens place of either the minuend or difference, although this must have required thinking from them. The teacher asked if these answers were correct by making them form another number sentence based on their answers. He commented that all their work were correct although their solutions were different.

Presented on the board, were two different correct answers. The teacher called the attention of the class to this by saying: "Look at this number and this number. Is there a difference?" This was to show that for this problem, they had more than one correct answer. In ending the lesson, he asked how many digits were missing in the number sentence in Problem 1. What he actually referred to was the number of missing digits in the tens column. However, this was not clear to the pupils. He also asked for the number of missing digits in the tens place of the number sentence in Problem 2. Although he had no time to elaborate because the class was over, the teacher was attempting to help the pupils recognize a relationship. There was only one correct answer in Problem 1 because there was only one missing digit in the tens column of the number sentence. This meant that two digits were known, so the third digit could be determined using the given operation. In Problem 2, there was more than one correct answer because there were two missing digits in the tens column of the number sentence. This meant that only one digit was known so another digit could be chosen, to get a third number. And there could be more than one choice.

REFLECTING ON NEEDED IMPROVEMENTS

In doing some things for the first time to engage his pupils in mathematical thinking, the teacher showed some good qualities. With commitment, he actively participated in developing an interesting and challenging lesson. In teaching, as much as possible he drew out from the pupils the correct responses by asking them questions. He also gave them enough time for problem solving. But because he was still adjusting, there are still aspects in his teaching that could be improved.

Processing of Pupils' Responses

Proper processing of pupils' responses is very important. The teacher needed to pay more attention to the correctness of their responses given orally or in writing on the board. He missed instances that could have been used to deepen their understanding of mathematics and made them think more deeply. An instance was the misreading of the numbers with missing digits. If he had caught the error and asked them to read correctly, then the number sentence for Problem 2 could have been written correctly. But given what had happened, he should have asked them if they noticed the mistake and that if such happens again next time, then they should ask him. It is important for pupils to have a questioning attitude. Another occasion was the writing on the board of incorrect answers such as the incorrect number sentence for Problem 2. If the error was identified at once, then more time could have been used for exploration in Problem 2 so pupils could possibly observe patterns and make conjectures that are very important in mathematical thinking (Stacey 2006). Another instance also was a boy's giving of the correct answer that the bigger number should be written above the smaller number. He could have probed for the reason for it and used it to lead the class to his expected reasoning. And still another instance was a girl's giving of a number sentence that he did not expect.

He expected to get:

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 6_ 5_4
-18 _9_
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 _ 3 7_8
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But the girl wrote:

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 18_ 9_
+ _37_8
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 6_5_4
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This could have been an opportunity to show that even if different number sentences were formed, the different correct answers could still be obtained.

Asking pupils to explain their work is another aspect that needs improvement. The pupils must have done some reasoning to find each missing digit in a number. But when they were asked to explain their work, some simply described how they used subtraction with regrouping in their number sentence. They did not give the reasons on how they found the missing digits in a similar way that the teacher had explained

when he worked on the problem in lesson planning. Through questioning, he could have enabled them to explain similarly. This could provide a conceptual meaning to procedures that is important in mathematical thinking.

Giving feedback on the correctness of a pupil's answer by the teacher's probing into his/her response or asking the class to comment on his/her work is also important. Without this, pupils would not know how to move on. This happened to a pupil who was called twice but because he did not know if his first answer was correct, he spent much time checking his second answer that depended on the first. The teacher should also have noted if pupils were careful to use the correct mathematical symbols for their work to be meaningful.

Additionally, board work has to be improved. Writing should be organized and should not be erased to provide a sequential record of what transpired in the lesson and help summarize important points (Wang-Iverson & Yoshida, 2005). The teacher could have guided the pupils in organizing their board work.

Helping Pupils Make Connections

There were several opportunities to emphasize relationships so that pupils could view the lesson coherently. After the missing digits in every number had been determined, the pupils could have been asked to interpret what those numbers were by relating them to the original problem situation. As it was, the pupils simply looked for the missing digits in every number. If the teacher had provided for this, then those who did not put the peso sign would have realized that their answers were meaningless.

Connections could have been made also by focusing on the features of the two problems through asking pupils to compare their similarities and differences. That the second problem could have many different correct answers that could be obtained in many different ways could have stood out if the teacher had asked the pupils to tell what they had observed about the two number sentences for the two problems. He could have done this when the pupils seemed to consider that only one digit could be correctly placed in the blank for the tens digit of the minuend in the second problem.

It is also important for the teacher to involve the whole class in analysing pupils' work. He could have asked them to compare their work and reasoning with those presented on the board and explained in class and to give comments about them.

Making the Most out of Pupils' Work

By consciously spotting the different answers of the pupils while they were working and later calling them to show and explain their work on the board, the teacher could have gathered more different answers and solutions that represented different ways of

thinking. From these, he could have asked what they observed about the answers and why they are different but still correct. However, he only called those who raised their hands and so it seemed that the same pupils were answering. He should also have seen if the pupils were really discussing while they were solving the problems.

CONCLUSIONS

With commitment and courage and having engaged in mathematical thinking, the teachers developed and taught a lesson that they had not taught before and in ways that they have done for the first time. They were certainly trying to adjust. They had done the best that they could to introduce problems that would develop pupils' mathematical thinking. Despite certain teaching aspects that need improvements, pupils' responses indicated that they were capable of engaging in mathematical thinking. When the teachers become more at home with their changed practices and engage more in mathematical thinking, possibly the pupils could engage better in mathematical thinking.

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