# MATHEMATICAL THINKING IN MULTIPLICATION IN HONG KONG SCHOOLS

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## Introduction

The development of multiplication is a progress that allows students to learn a lot of patterns and also mathematics structures. This paper is based on a research of classroom teaching in mathematics of primary 3 to primary 5 students in Hong Kong. The same set of questions used in a primary 3 to primary 5 classes (except the question on Pissa). With the design of different sets of worksheets on non-routine problems, the development of concept of counting, repeated counting, multiple with counting, using multiple and then the law of multiplication to solve the problems is discussed. The results inform us that children using multiplication as a tool to solve questions of combinatoric nature (such as number of different grid formed by using different number of colour etc), are more difficult to understand than we thought. And the jumping of the cognitive gap from repeated counting and addition to multiplication needs certain daily examples to act as correspondence in concepts formation.

The using of the designed worksheets helps students to develop their concept of multiplication and also mathematical thinking through the connection of concepts by overcoming cognitive gaps.

## The set of question used

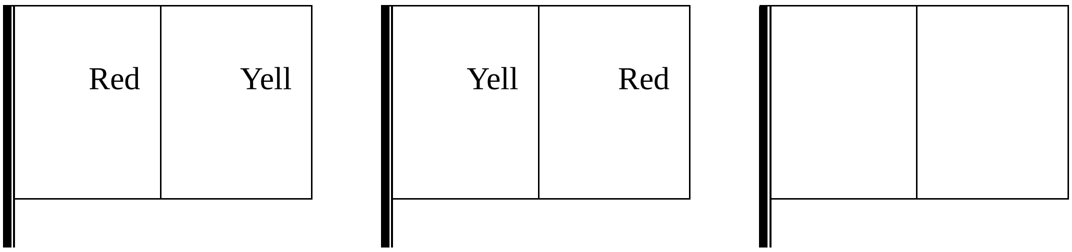
SET 1

| |
|---|
| Question<br>A flag has 2-grid, if you have k colours to fill in the grids, and only each one of the colour is used once.<br>How many different way are possible? |

The worksheet has the following smaller questions. This is the format of all the sets of papers.

| | |
|---|---|
| 1 | A flag has 2-grid, only each of the colours red and yellow can be used once.<br>How many different way are possible? |

| | |
|---|---|
| 2 | A flag has 2-grid, only each of the colours red and yellow can be used once.<br>How many different way are possible? |
| 3 | A flag has 2-grid, only each of the FOUR colours can be used once.<br>How many different way are possible? |




Students can find out that using FOUR colours can give them 12 different ways.

| A | B | C | D | E |
|---|---|---|---|---|
| AB | BA | CA | DA | EA |
| AC | BC | CB | DB | EB |
| AD | BD | CD | DC | EC |
| AE | BE | CE | DE | ED |
| 5 | 5 | 5 | 5 | 5 |

By listing the table, some students are able to use the multiplication 5×4 = 20 to obtain the answer.
This is a systematic counting.
Can students jump from the results of 5 colours to 6 colours?
Many students still relay on the listing of the table for using SIX colours.

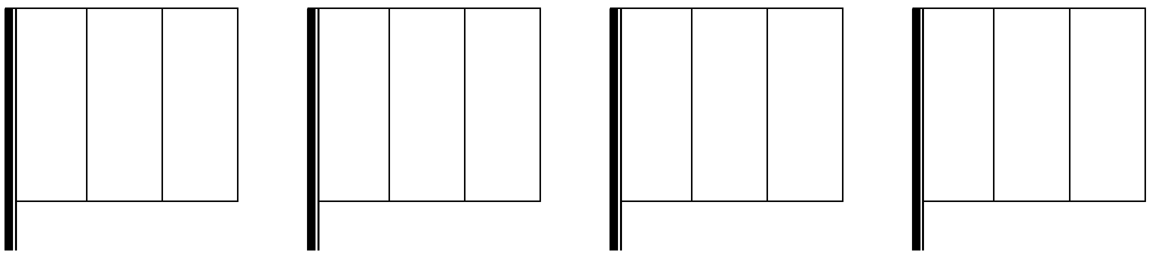
For primary 5 students, many can see that they have (k-1) choices for the first colour and k choices for the second colour. This is interesting that they think in the following process. If repeating the colours are allowed, for 7 colours, there will be 7×7 = 49 ways, but the answer is “(7-1)×7 = 42, for it is not allowed to repeat the colours”. They use (7-1)×7 = 42 rather than 7×(7-1) = 42. The same for the case of EIGHT colour, (8-1)×8 = 56 and not 8×(8-1) = 56.

| **Summary on 2-grids：** | |
|---|---|
| **Number of Colour** | **Number of ways** |
| **2** | |
| **3** | |
| **4** | |
| **5** | |
| **6** | |
| **7** | |
| **8** | |
| **k** | |

It is very difficult for students to generalize the case to “k” colours. Even for the primary 5 students, they are not able to see that the answer is (k-1)×k.
However, given a large value of k, say k = 100, some students can provide the answer of 100×99.

SET 2 using of 3 grids and k colours.

> Question
> A flag has 3-grid, if you have k colours to fill in the grids, and only each one of the colour is used once.
> How many different way are possible?



Since there are three grids, the listing is more difficult than the worksheet in SET 1. Students start to use A, B, C to represent colours after some hints and they use listing to count the answer. This is difficult to use table and ABC to count 4 colours and the following is how some students tackle the problem.

For FOUR colours (A, B, C, D), the first colour A can give 6 options

| A | B | C | D |
|---|---|---|---|
| ABC<br>ACB<br>ABD<br>ADB<br>ACD<br>ADC | | | |
| 6 | 6 | 6 | 6 |

Thy use counting for the first colour (A), and then they know that it will be the same for the other colour. Hence they got the answer $6 + 6 + 6 + 6 = 24$.

And a while later, students formulate the results for $6\times4 = 24$.

Similarly, they do the same for FIVE colours in 3-grids. But still students could not obtain the relationship of $5\times4\times3 = 60$.

However, many can fill in the following table up to case 8.

| Summary on 3-grids : | |
|---|---|
| Number of Colour | Number of ways |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| k | |

SET 3 Using number to fill in the grids

| |
|---|
| Question<br>There are 3-grids. Insert the number 1, 2, 3, …, k into the grids once.<br>How many different ways are possible? |

The first question is using three numbers 1, 2, 3.

| | | |
|---|---|---|
| 1 | 2 | 3 |

| | | |
|---|---|---|
| 1 | 3 | 2 |

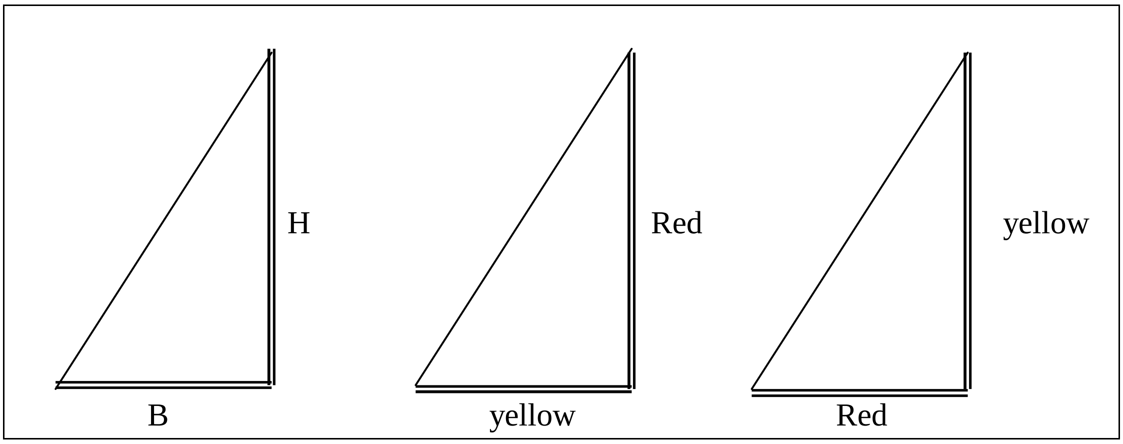
This set of papers is used in another class of primary 4. It is interesting that this problem, which does not have the context of a flag (grid), allow students to be more focus on the multiplication part.

The process of solution is similar. They list the cases and count the number. More students can obtain the results more quickly in compare to the answer in SET 2.

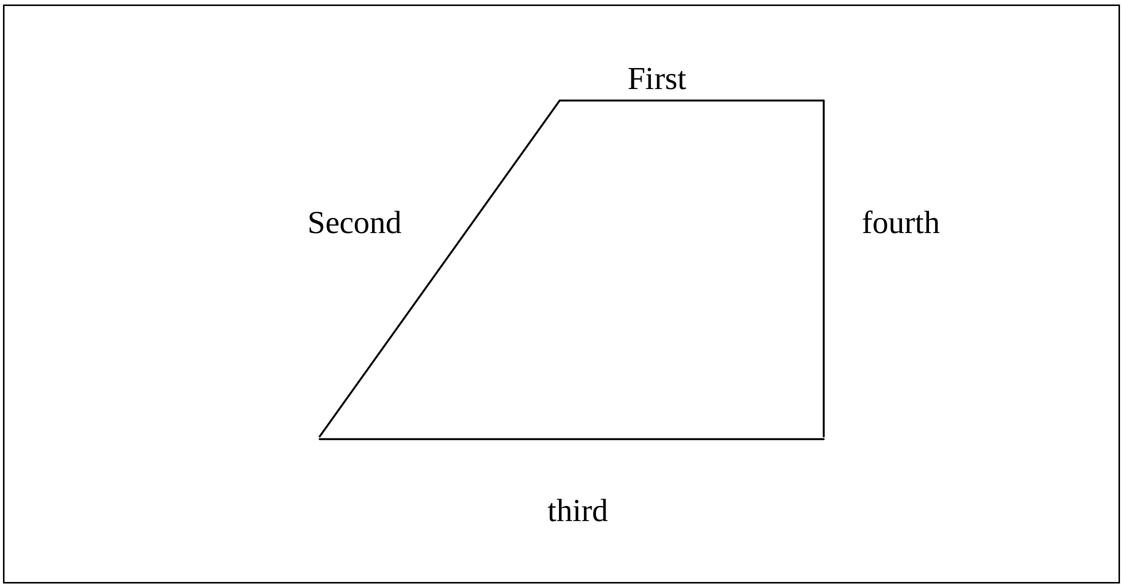
SET 4 Using a context of triangle

> Question
> If you can only use the colours “red”, “yellow” and “blue” for the triangle, each side uses only one colour, how many different ways are possible?

Students use table to obtain the answer quickly for triangle and quadrilateral.




| Ways | Base colour | Height colour |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |




| Quadrilateral | |
|---|---|
| Number of colour | Different number |
| 4 | 24 |
| 5 | 120 |
| 6 | 360 |
| 7 | 840 |
| 8 | 1680 |
| 9 | 3024 |
| k | (k-1)(k-2)(k-3)k |

However, for pentagon, no primary 5 students can obtain the general for of the result (k-1)(k-2)(k-3)(k-4)k.

| Pentagon | |
|---|---|
| Number of colour | Different number |
| 5 | 120 |
| 6 | |
| 7 | |
| k | (k-1)(k-2)(k-3)(k-4)k |

SET 5 Pissa toppings

| |
|---|
| Question<br>You can choose a number of toppings for a pissa.<br>How many different kind of pissa are possible if there are k different choices toppings? |

| ways | Toppings | | |
|---|---|---|---|
| | Cheese | Beacon | Sausage |
| | | | |
| | | | |

For the case of two to three toppings, students can obtain the answer easily. They do that by counting. Students can use counting to handle the question up to 4 choices of serving. For 5 or more toppings, students' counting can be lost and correct answer can only be obtained through the using of multiplication.

For 4 toppings, many students can get 16 ways. Their strategy is by using ticks in the table.

| Toppings | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| | ✓ | ✓ | ✓ | ✓ |
| | ✓ | ✓ | ✓ | ✗ |
| | ✓ | ✓ | ✗ | ✗ |
| | | | | |
| | | | | |

Both primary three to primary five students use counting to solve the problem. However, after using three toppings, the counting process is tedious. For example, students make mistake in using four toppings and give out answer such as 15 or 12. They have missed some cases in their counting.

Those who counted correctly try to use the pattern of the first few answer to obtain the solutions. Students do think that answer must satisfy the pattern they discovered, that is, in the sequence of 2, 4, 8, 16, 32 etc.

The first thinking process is a recurrence relation. If there are 4 ways for two toppings, and when one more topping is added, then there are two possibilities for the case of two toppings. The first one is no third topping is needed in this 4 cases, and the second one is the adding of the third toppings for the previous 4 ways. So students based on 4 + 4 = 8 to get their answer and not based on $2^3 = 8$.

For example, the following is the answer of two toppings, 4 ways.

| | A | B |
|---|---|---|
| 1 | ✓ | ✓ |
| 2 | ✓ | × |
| 3 | × | ✓ |
| 4 | × | × |

Then they repeated the above pattern, adding the third toppings, by giving four "✓" and four "×"., getting the answer 4 + 4 = 8.

| | A | B | C |
|---|---|---|---|
| 1A | ✓ | ✓ | ✓ |
| 2A | ✓ | × | ✓ |
| 3A | × | ✓ | ✓ |
| 4A | × | × | ✓ |
| 1B | ✓ | ✓ | × |
| 2B | ✓ | × | × |
| 3B | × | ✓ | × |
| 4B | × | × | × |

They need the answer of two toppings to obtain the answer of 4 toppings, and the answer of 5 toppings to get their answer of 6 toppings etc. They know that the "second" answer is a double of the first answer. However, they could not directly get the answer of 6 toppings from multiplication.

However, some students found that they can formulate question into either taking each of the toppings or not taking the toppings. There are two choices for each topping and their results are summarized in the following table.

| Toppings | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Yes | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| No | × | × | × | × | × | × |
| Number | 2 | 2 | 2 | 2 | 2 | 2 |

So he found that the answer is $2\times2\times2\times2\times2\times2 = 64$.
The totally abstract use of multiplication is based on the understanding of the problem, familiar with the multiplication table and also the relation of repeated addition.

**Summary**

The above teachings show that it is difficult for students to formulate the solutions through multiplication. The thinking process of addition, repeated addition and multiplication need certain space for students to overcome their cognitive gap. Through such discussion of the listing of table, students can think of different ways in solving the problem. Though some of their answer through counting is not correct, they can verify them after some classroom discussion.