REASONING ABOUT THREE-EIGHTHS: FROM PARTITIONED FRACTIONS TOWARDS QUANTITY FRACTIONS

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Curriculum documents describe the importance of questioning, reasoning and reflecting as contributing to Working Mathematically. A research lesson on the development of units of different sizes (eighths) associated with measurement and fractions, is developed as a vehicle for developing mathematical reasoning through argumentation in a composite Year 3–4 class. Making a transition from embodied fractions (parts of a whole) to recognising the equal whole needed for comparison of fractions as mathematical objects extends the current Mathematics curriculum in New South Wales. The lesson study also highlighted the need to develop taken as shared meaningfor fraction units used in classroom argumentation.

REPRESENTATIONAL CONTEXTS

Working Mathematically draws on ways on ways of seeing, questioning, interpreting, reasoning and communicating. This type of mathematical thinking was summarised by Schoenfeld as follows:

> Learning to think mathematically means (a) developing a mathematical point of view— valuing the process of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of understanding structure—mathematical sense-making. (Schoenfeld, 1989, as cited in Ball, 1993, p. 157)

The tools used in the service of understanding structure are often derived from *models of* the mathematics. For example, partitioned circles or rectangles are used as regional models of partitioned fractions (Watanabe, 2002) which can contribute to associated concept images of fractions . Yet physical models have limitations in that they at best imply the mathematical concept, and can add unwarranted components to the intended concept image. In particular, students' evoked fraction concepts suggest that equality of area is not always the feature abstracted from regional models used in teaching fractions (Gould, 2005). Further, to be able to interpret the part -whole comparison of area intended by the regional model, children need to be familiar with the context, which in this example includes the concept of area (Lampert, 1989). Choosing an appropriate representational domain is an important teaching decision in developing students' mathematical understanding.

In practice, fractions exist in essentially two forms: embodied representations of comparisons, sometimes called partitioned fractions, and mathematical objects, also known as quantity fractions. A *partitioned fraction* (Yoshida, 2004) can be described as the fraction formed when partitioning objects into *b* equal parts and selecting *a* out of *b* parts to arrive at the partitioned fraction *a/b*. A partitioned fraction can be of either discrete or continuous objects but a partitioned fraction is always a fraction of something. By comparison, quantity fractions are mathematical objects defined as fractions that refer to a universal unit. Asking the question, which is larger, one-half or three-eighths, only makes sense if the question is one of quantity fractions. Quantity fractions implicitly reference a universal unit, a unique unit-whole, which is independent of any situation. If one-half and three-eighths as mathematical objects do not refer to a universal whole, we cannot compare them.

The learning of fractions is subject to a paradox that is central to mathematical thinking (Lehrer & Lesh, 2003). On one hand, a fraction such as 2/3 takes its meaning from the situations to which it refers (partitioned fraction); on the other hand, it derives its mathematical power by divorcing itself from those situations (quantity fraction). Working with partitioned embodiments of the fraction “3/7” can elicit a parts-of-a-whole meaning as “three out of seven”, but without divorcing the fraction notation from this context interpreting “7/3” does not make sense. It is difficult for students to become aware of a unit-whole when the unit-whole is often implicit in everyday situations involving fractions. To make the transition from partitioned fractions to quantity fractions, students need to develop a sense of the size of fractions. A sense of the size of fractions is what Saenz -Ludlow (1994) refers to as conceptualising fractions as quantities. Mathematical thinking associated with working with units of various types and in particular, the introduction of abstract units, are central to mathematics.

PLANNING THE LESSON

In the dominant instructional model used to teach fractions in New South Wales, children learn to divide objects into equal parts. Next, children learn to count the number of parts of interest and place the result of this count above the count of the total number of parts using the standard fraction notation. This part-whole recording method is used to introduce fraction notation. The transition from counting parts of a model to recording fraction notation is followed by instruction on the traditional algorithmic manipulation of the whole number components of the fraction notation, known as operating with fractions.

Unfortunately, the predominant instructional model is not successful for many students and fractions are a particularly troublesome area of the elementary mathematics curriculum in NSW. The language associated with fractions in English contributes to a number of misconceptions for students. Unlike most Asian languages, English uses the same terms for naming ordinals and fractions (e.g. third, sixth, ninth). It is also easy for students to not hear the soft sounds at the end of fraction names, which can lead to confusion between whole numbers (e.g. six) and fractions (e.g. sixth). Thus, although six sixes are thirty -six, six sixths are one.

Following the mathematics syllabus, children describe halves in everyday contexts in their first year of school, Kindergarten. Ironically, everyday representational contexts for halves include examples such as cutting a piece of fruit into halves, where the means of determining the equality of the pieces relies on an understanding of volume. In the following two years (Years 1 and 2) children are expected to model and describe a half or a quarter of a whole object or collection of objects as well as to use the fraction notation 1/2 and 1/4. In Years 3 and 4 children model, compare and represent fractions with denominators 2, 4 and 8 as well as find equivalence between halves, quarters and eighths.

For fractions to make the transition from embodied partitions to mathematical objects the idea of a universal unit-whole needs to be established. This universal unit-whole is a ‘one’ that remains the same size in all contexts and is similar to a standard unit of measure. Making a transition from embodied fractions (parts of a whole) to recognising the equal whole needed for comparison of fractions as mathematical objects is the unit goal for lesson study outlined below. The idea of lesson study was new to the teacher and the school. Further, the idea of the need to identify the equal whole and the specific role of representational contexts are not part of current teaching practice in elementary schools in New South Wales. Consequently, the planned abstraction referred to in the unit goal is a very ambitious goal for the composite Year 3 and Year 4 class taking part in the lesson study.

Developing thinking through argumentation

The key window for considering mathematical thinking in this lesson study is through justification in reasoned argument. Learning to argue about mathematical ideas is fundamental to understanding mathematics. Palincsar and Brown (1984) wrote that “ ...understanding is more likely to occur when a child is required to explain, elaborate, or defend his position to others; the burden of explanation is often the push needed to make him or her evaluate, integrate and elaborate knowledge in new ways.” Argument here is taken to mean a discursive exchange among participants for the purpose of convincing others through the use of mathematical modes of thought.

The ways in which students seek to justify claims, convince their classmates and teacher, and participate in the collective development of publicly accepted mathematical knowledge contribute to mathematical argument. In a culture that expects student understanding, teaching mathematics is more than merely telling or showing students; teachers must enable students to create meanings through their own thinking and reasoning. Classroom argumentation needs opportunities to move from authority-based arguments (because the teacher says so or the text states this) to reasoning with mathematical backing. That is, “how do you know?” is the key question. The expectation is that students arrive at consensus through reasoned argument, reconciling different approaches through demonstration using a common model.

THE LESSON: THREE-EIGHTHS OF THE BOARD

The link between the process of division and the creation of fractions is not always clear to students. To simplify the creation of this link, the attribute of length is used instead of area to create partitioned fractions. Although regional models are often used to introduce fractions, some students focus on the number of regions and not the area of the regions compared to the whole shape in abstracting the fraction relationship.

By the start of Year 3, children can model and describe a half or a quarter of a whole object or collection of objects as well as being familiar with the fraction notation 1/2 and 1/4 (syllabus reference NS 1.4). Multigrade classes are quite common in New South Wales and the class taking part in the lesson study described here had 6 Year 3 students and 21 Year 4 students. The Mathematics K–6 syllabus describes content in stages corresponding to two school years with the exception of the Kindergarten year, referred to as Early Stage 1. As the students in the study had covered the fractions content from Stage 1 (Years 1 and 2) this lesson was designed as an introduction to eighths and the relationship between eighths, quarters and half (Appendix A).

The lesson started by inviting three students to estimate where three-eighths of the width of the class white-board would be, mark the point and put their initials next to their estimates. The students were chosen by the teacher based on her knowledge of the students to obtain variation in the estimates. The fraction 3/8 was used with the attribute of length to focus on composition of partitioning through repeated halving. As the students had previously covered work on 1/2 and ¼ the use of 3/8 also provided scope for iterating the unit fraction. Having the students record their initials next to the estimates gains ownership of the estimate by the student and encourages a desire to find out. It also makes discussion about the estimates easier by providing a name for the estimate.

A piece of string the same length as the white-board was cut and used by students to form one-eighth and in justifying which estimate was closest. The class discussion also provided opportunities to relate the values of half, quarters and eighths of the same length. The students returned to their desks following the discussion and located positions corresponding to various numbers of eighths of different sized intervals (see Appendix B). After discussing the location of three-eighths on different sized intervals the final question was posed: *Could 4/8 ever be less than 2/8?*

Students' explanations

Having established the unit of one -eighth, the teacher used one -half of the length of the whiteboard as a benchmark to determine how close the estimates were to three-eighths. The teacher asks how close was Jack's estimate to three-eighths and a very rapid exchange takes place between the students. One student (Charlotte) says that Jack was not as close as Emily, a mark greater than one-half of the length of the board. Another student thinks about the response and quickly says, "That's a half". The teacher picks up on the exchange.

Teacher: *That’s an interesting comment. Charlotte’s comment was he wasn’t as close as Emily.*

Teacher: *Remember we were wanting to find three-eighths of the length of the white-board. If this is halfway, how much of the white-board do you think Emily found?*

Stephen: *Two, three quarters, two and a half quarters.*

Teacher: *Two and a half quarters? OK. Amy, how...*

Amy: *Four and a half eighths.*

Teacher: *Jessica.*

Jessica: *It could be five-eighths.*

Jessica then went on to describe the size of the unit (partitioned) fraction one-eighth and that one half of the board corresponds to four-eighths (Fig. 1), so adding on one -eighth more making five-eighths would be a position very close to Emily’s mark.

Figure 1. Describing one-half as four-eighths

The discussion arising from using the benchmark of one-half to determine which of the estimates was closest to three-eighths, and the subsequent description of each estimate location in terms of eighths, was very informative. The orchestration of the discussion did enable the teacher to intervene to seek justifications for the different beliefs as to which estimate was closest to three-eighths.

However, developing *taken as shared* meaning for fractions as quantities is not simple. In a later part of the lesson, when one student explains why three-eighths is at a different location on the first two intervals, he suggests that it might be because “all of us used different strategies to work out the answer”. This suggested was followed by a surprising exchange.

Teacher: *What strategies did you use?*

Student: *I cutted them up into all kinds of different quarters and then I went to... uhm, and then I counted it from one for every quarter.*

Teacher: *Why did you cut it up into quart ers?*

Student: *Because ... uhm...* [Pause]É *I cut it up into different quarters because then I’d know how much of each part of it is the same.*

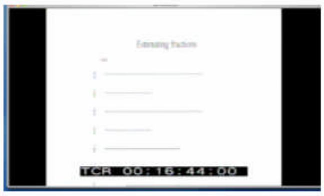
The exchange is puzzling for both the student and the teacher, as they appear to have different meanings for quarters. When students are challenged to explain their reasoning the evoked concept images (Tall & Vinner, 1981) can reveal unexpected interpretations of fractions such as quarters. The evoked image related to the term quarter was quite different between student and teacher. The student was using the term quarter for a general fraction part. Rather than meaning one of four equal parts comprising the whole, the term quarter meant 'equal piece' for this student. This interpretation was confirmed when the video of the lesson was replayed to the student. Just as the common use of fraction terms can lead to exchanges that suggest that one-half can be bigger than another half (*You take the bigger half)* the same appears to occur with the term quarter. As the teacher and the student did not have a *taken as shared meaning* for a 'quarter' the discussion did not advance the understanding of fraction units. Recognising that the nature of mathematical argument may vary between cultures (Sekiguchi & Miyazaki, 2000) confronting the individual's misconception about fraction units was not always straightforward. The teacher can only respect a student's idea if the teacher and the other students can understand the idea. Although the teacher encourages other students to think about how quarters might be of use in determining three-eighths, the problem cannot be truly shared if individuals hold different concept images for quarters.

REFINING THE LESSON

Although the lesson was generally quite effective at encouraging students' justification of fraction units and in preparing to make a transition from embodied fractions (parts of a whole) to recognising the equal whole needed for comparison of fractions as mathematical objects, revisions are needed to better address two areas. The first area relates to the number of students who reached the desired goal of the lesson. About one-quarter of the class had difficulty generalising the process of repeated halving to create eighths of different sized units. This suggests that it might be helpful to carry out the process of finding three-eighths of a different length, estimating and then using repeated halving, with students clarifying the process before moving to the worksheet. The second area, which is integrally related, is developing *taken as shared meanings* for one-quarter and one-half. Discussion of the strategies used to partition intervals into eighths relied on shared meanings for quarters that did not appear to exist within the class. The next lesson in this unit looked at students partitioning pieces of string using small pegs to construct related fractions. This activity (related to eighths, quarters and halves) may have helped in creating the *taken as shared meanings* for these fractions before discussion arising from the worksheet activity (Appendix B).

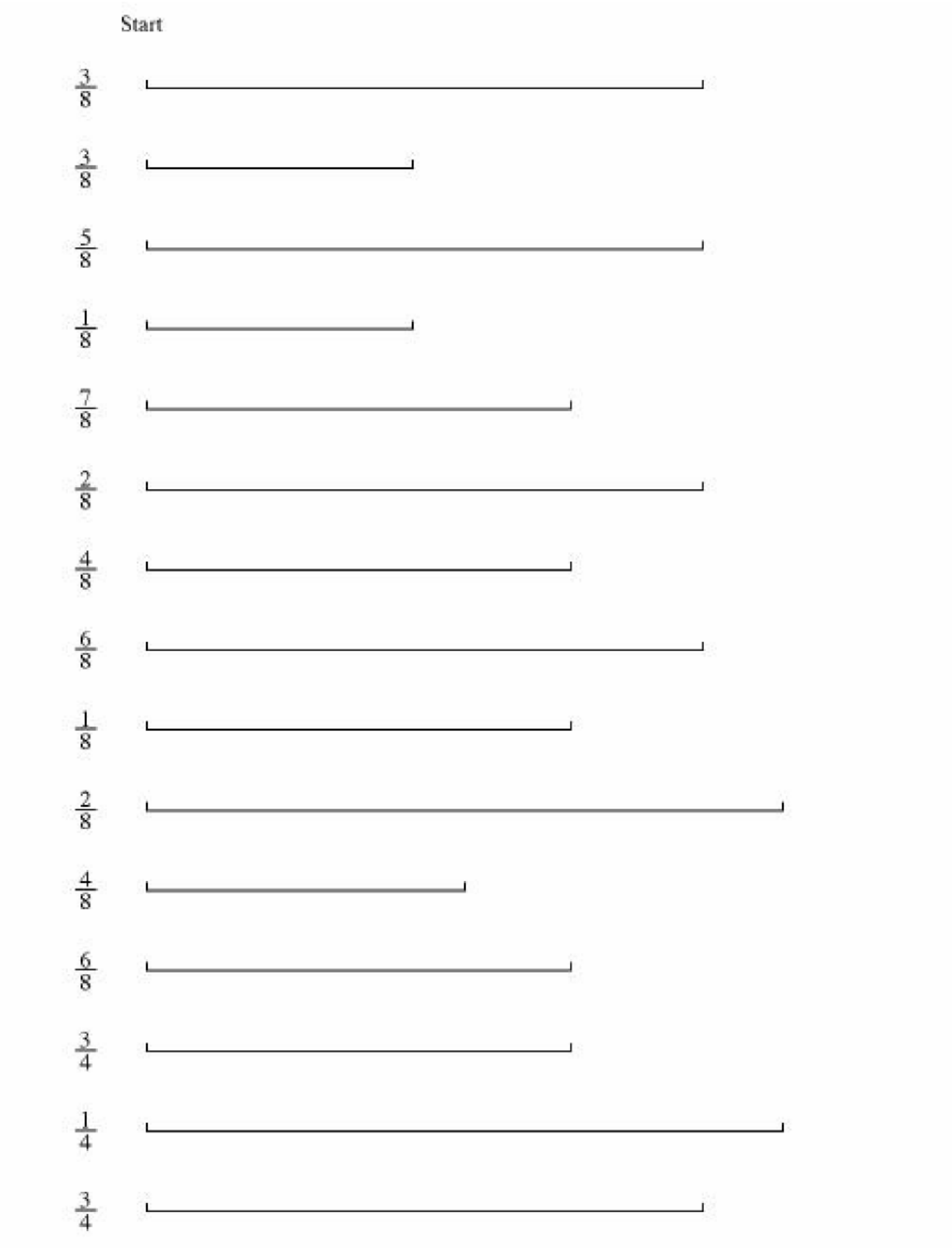
APPENDIX A: ESTIMATING 3/8 OF THE WIDTH OF THE BOARD (GRADE 3-4)

Goal: The fraction 3/8 is used with the attribute of length to focus on composition of partitioning through repeated halving, to link to students' concept images of one-half and one-quarter.

	OUTLINE	COMMENTS
Problem posing	Challenge the students to estimate 3/8 of the distance from the left side of the board. Invite three students to mark the distance and to add their initials to the marks they make. T: *How could we determine which mark is closest?* S1: *Measure the distance, divide by eight and multiply by three.* S2: *Find half way, then find half of that to get a quarter, then find half of the distance between one-half and one-quarter.* S3: *Get a piece of string the length of the board and then fold it into eight.*	If the students have difficulty with 3/8 change the question to estimating 1/8 first. Repeated halving is the method used by most students. 
Justifying solution methods	Discuss the strategies. Using a piece of string, invite a student to demonstrate how to check which estimate is closest and justify his or her reasoning. Compare the location of 3/8 and half way. Determine the location of the three crosses in terms of eighths. Use a worksheet involving a number of intervals of different lengths that require students to locate fractions relating to different numbers of eighths (e.g. 3/8, 5/8, 7/8, 1/8) NS2.4 The first two questions ask students to estimate the location of three-eighths on different sized intervals. Check to see if students are locating three-eighths at the same distance from the left, and discuss. *Should the answers be in the same place?*	 Check understanding of the number of folds and the number of parts. 
Comparing units	Put the following question on the board. *Could 4/8 ever be less than 2/8?* Use think-pair-share or a quick feedback back method (e.g. thumbs up, down, sideways) to determine student responses.	This question is designed to prompt the introduction of the need for the equal whole in the transition from partitioned fractions (fractions of things) to quantity fractions (fractions as numbers).

Appendix B

Estimating fractions



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