HOW TO DEVELOP MATHEMATICAL THINKING

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1. Thinking mathematically and mathematical thinking

(1) Idea of Mr. Kenzo Nakajima

Who introduced mathematical thinking into the Course of Study revised in 1958
as aims of mathematics in Japan. Creative activities to be good for mathematics nearly equal to 'thinking mathematically' Mathematical thinking in the Course of Study revised in 1958; aims of elementary school mathematics.
In the aims, mathematical thinking located in the two phases

Mathematical thinking as results created by students
Mathematical thinking as tools students use adequately

There was developing of a scientific attitude in the background.
Mathematical thinking as one point of view of evaluation (after 1970's)

→ development of mathematical thinking
→ meaning of 'development' became ambiguous
→ need to realize the two phases of mathematical thinking again

(2) Idea of Mr. Shigeo Katagiri

His life work is to analyze and classify mathematical thinking from 1960's
Mathematical thinking consists of mathematical idea, method, and attitude which support thinking mathematically

2. Developing student's mathematical thinking in classroom

(1) Putting student's activities in the center of classroom and these activities to be creative or inventive for students

A lesson (classroom) develops mathematical thinking by students' problem solving. Teachers guide and support their activities.

(2) Creative activities (problem solving) should be meaningful both for students and teachers.

We try to analyze the elements and structures of mathematical thinking and to help students acquire them.

3. Thinking mathematically

(1) Students' independent activities

Engaging oneself, not other people's activities

(2) Motivations and phases of activities

Engaging mathematical activities according to phases of them adequately

Motivations and phases

・ from Need in life, Explanation of phenomena

→ Using mathematics

Considering or judging by using mathematics

・ from Intellectual curiosity, Pursuit of mathematical beauty

→ Creating or discovering mathematics

Thinking creatively or extensively and discovering or inventing new facts, skills, ideas etc. By relation with mathematics of experience by everyday life and having learned already.

・ Supporting using and creating mathematics, and from pursuit conclusive evidence or enrichment

→ Explanation or verification

Necessity for understanding by oneself, persuading other peoples sharing results each other, and refining them better

4. Observing classroom activities

5th grade sum of interior angles of polygons

5. Some points of view for improving math classes

(1) To help students make thinking mathematically a custom
(2) To represent students' inner process of thinking mathematically
(3) Grasp results exactly from thinking mathematically or mathematical problem solving
(4) Two adequacy for posing problem

- to be good for aims of lesson
- to lead results to be good for problem posed

(5) Problem posing and the result of solving the problem posed

- to consider the characteristic of problem; self-creating aspect

(6) Developing the mind of challenge, confidence, feeling of effectiveness
(7) Collaboration and creating
→ Japanese proverb; sann-ninn yore-ba monnjyu –no chie

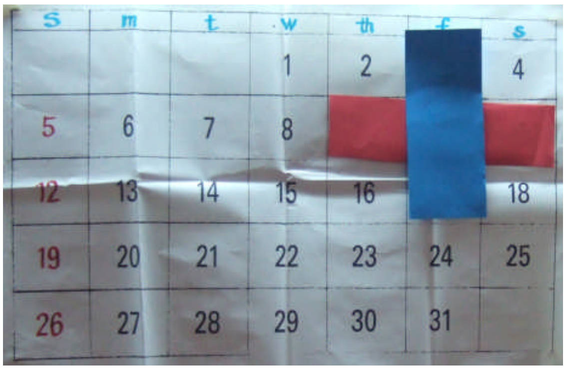
How to develop Student's Mathematical Thinking in Classroom

Objective: 1) To find and think about average 2) To develop mathematical thinking and children's image from surroundings.

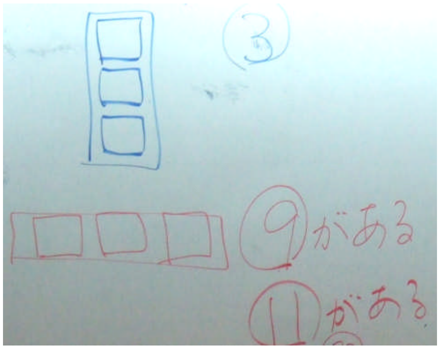
Field note：

13:24～
T This is Japan's map. Around country is sea, I will give this to your teacher. Also these color papers. I will give you a card with Japan's view.

13:30
T. I will stick this paper. Watch it carefully.
  What it is?
C. Calendar
S. How to say Sunday?
C. Sunday…Saturday
T. I will hide somewhere.
C. It looks like Hospital sign.

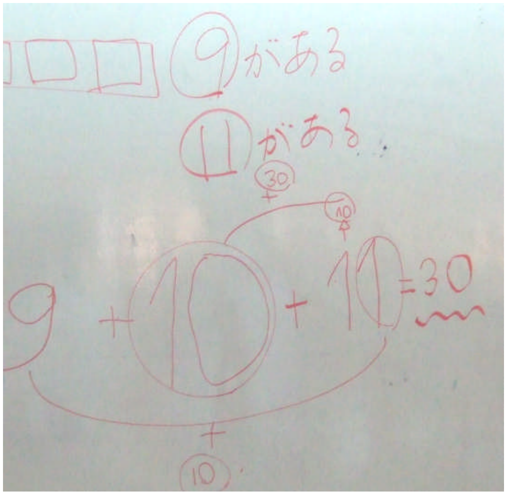


T. Summary of Blue is bigger or Red is bigger?
Blue 3+7+14 Red 6+7+8
Almost C. answer Red is bigger than Blue



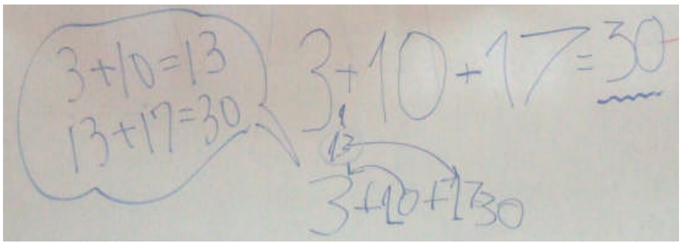
T. Why do you think so?
C. Cause it has 9 so it's bigger
T. It has 3 numbers
T. I will ask you again. Summary of Blue is bigger or Red is bigger?
  Nobody answered Blue is bigger
C. Because Blue has 3, but red has 11
T. We will check it by calculate them

T. Had already learnt addition right?
T. Blue is 3+10+17 right?
T. Red is 3+10+17 right?
C. Wrong!!
T. Can you correct it?
C. 9+10+11
T. What is the answer?

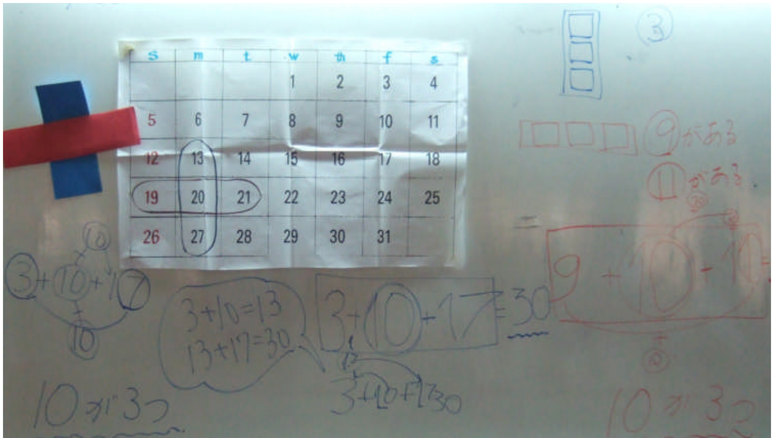


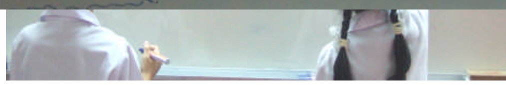
C. Blue 30.Clap. Red 30.clap
T. Blue is 30. Red is 30. They are same.
T. How do you calculate the red one?
C. 9+11 is 20 then 20 + 10 is 30
T. Can you draw the lines?
T. Look it carefully. Do you have other way to calculate?
C. 3+10+17. 10+3 is 13 then 13+17 is 30

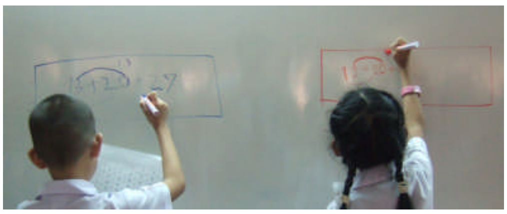
13:52
T. She/he calculates from left to right. But another he/she did the faster way to calculate. It is easier to calculate, when we have three of tens. Can we do like this in that one.
T. discuss with your friend
C. Draw the line 3 and 7 of 17
T. 10, 10, and 10 are 30
Both have there of tens



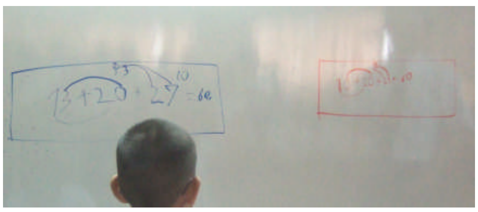
We can calculate faster if make there of tens.
10 is central, inside the card.
If we can find this middle number, we can calculate faster.
T. I'll change cards position.
What are summary of them?
Blue: 13+20+27 a lot
Red: 19+20+21 a little
Same: a little
T. Please. Calculate them.
C. Same
T. What it is?
C. 60
T. Writes the equation please.







14.02
T. It is really to be 60?



14.07

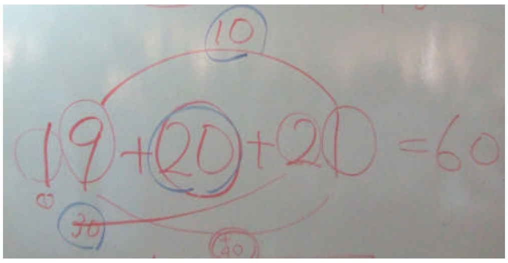
T. Is it possible to leave the central number?

C. Yes. 19+20+21, 19+1 is 20. 20+20+20

T. What about blue one.

C.13+20+27, 3+7 is 10. 20+10 is 30

T. Good Job! It faster to calculate when leaves central number.



14.12

T. If on the calendar, wherever we put cards it will be same?

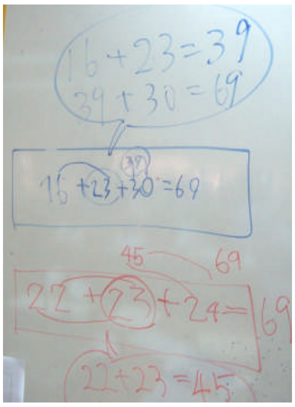
T. Anywhere you want to put cards?

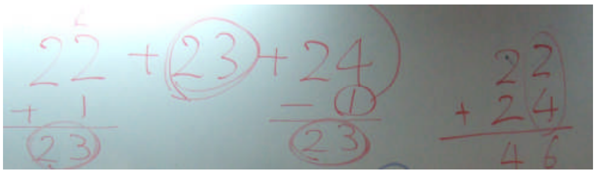
C. 16+23+30

C　赤 :

T　　同じになる:　違う :　 分から ない :

T. I will take cards off and circle around, instead.





WHERE DO MATHEMATICS PROBLEMS COME FROM IN ELEMENTARY SCHOOL CLASSROOMS?

The Problem Solving Approach in the classroom: a dialectic with the conflicts and comprehension among students

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The book (Isoda, 1996) was written to support elementary school teachers in Japan who plan lessons based on the Problem Solving Approach, which is a renowned approach for teaching mathematics around the world. According to mathematics education theory regarding lesson plan development or textbook sequencing, mathematics educators usually take account of the sequence of mathematical content, a range of situations including real-life examples, and mathematical representation for the process of abstraction. For example, one of those embedded in the textbook based on the 'Model of, Model for' framework by the Freudenthal Institute is 'Mathematics in Context', which includes the process of Situation, Model and Form through Mathematization.

Across the world there are different textbooks, based on the local curriculum. However, most of these textbooks do not directly deal with students' misunderstandings. On the other hand, Japanese elementary school textbooks and teachers' guide include expected children's answers for each problem and suggestions for how to treat children's misunderstandings in class based on the experiences of lesson study. This is possible because the Japanese curriculum is national and textbooks are shared. When curriculums vary across different schools and classrooms, it is not easy to share these kinds of ideas.

Many elementary school teachers and mathematic educators believe that mathematics problems arise from daily situations. Isoda (1996) discussed an alternative idea based on Japanese tradition that mathematics problems arise from problematic situations for children in special occasions during lessons that follow the curriculum sequence. In the Japanese Problem Solving Approach, known since the 1 960s, a problem situation is defined as confronting the unknown when compared to what has previously been learned. Problems posed within the teaching sequence, developed via curriculum planning, enable children to learn mathematics based on prior learning. What is theoretically new in this book (developed early 1990s), are the following. The book describes the source of problem situations through the process of extensions originated from the curriculum sequence; it explains the development of conceptual and procedural understanding through the learning of mathematics based on the curriculum sequence; and it likewise explains how to utilize dialectic discussion among classroom (Neriage), which involves each other's perspective.

Library data for your reference:

Masami ISODA edited (1996), Problem-Solving Approach with Diverse Ideas and Dialectic Discussions: Conflict and appreciation based on the conceptual and procedural knowledge, Tokyo:Meijitosyo Pub. (written in Japanese) (English Ver.6 for the Preparing Publication. Copyright ©Masami ISODA, CRICED, Univ. of Tsukuba. All right reserved)

Chapter 1
The Lesson Structure Based on a Problem-Solving Approach that Produces Diverse Ideas and Promotes Developmental Discussions: Focusing on the Gap between Meaning and Procedure

The lesson planning based on the theory of understanding on curriculum sequence

Masami Isoda, University of Tsukuba

In an introductory lesson on adding fractions with different denominators that aims to teach children how and why they should perform calculations like 1/2 + 1/3, children who do not know the meaning of 1/2 L (ℓ) or 1/3 L cannot objectively understand the meaning of the word problem. Children who are not proficient in the procedures of reducing fractions to a common denominator, previously learned, will likely struggle with solving problems. Teachers will be well aware of the importance of meanings and procedures (including form and way of drawing) learned over the course of problem-solving lessons.

The Japanese Problem Solving Approach usually begins from children's challenges of a big problem based on what they have already learned. This chapter will use specific examples to show that previously learned meanings and procedures (form and way of drawing) help elicit a variety of ideas (conception) from children. Then it will describe methods of creating lessons that support children's learning through the eliciting of diverse ideas (even if it is misunderstanding) and a developmental discussion (a dialectic among students). This is based on the notion that it is precisely when people are perplexed by something problematic that they develop their own questions or tasks, have a real opportunity to think about these, can promote their own learning, and can reach a point of understanding. The following aims to shed new light on the true significance of this notion.

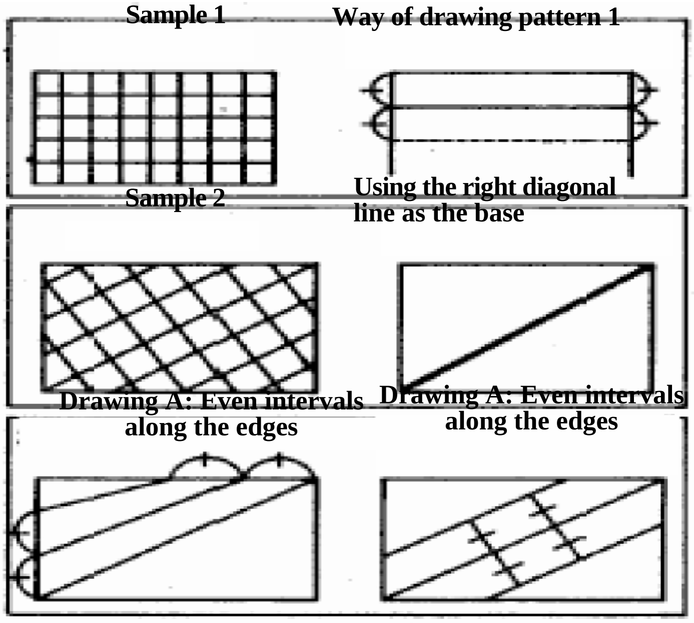
1. It goes well! It goes well!! What?

In Japan many teachers have experienced the following situation. The teacher finishes a class feeling confident that the lesson went well and believing that the children understood the material, but the children say "What? I don't understand" in the very next class. The student comments clearly indicate that they had not developed a good understanding of the material previously presented, even if they said they had clearly understood it at that time. This is precisely the treasure secret of the problem-solving approach: to elicit diverse ideas including misunderstanding and promote developmental discussions.

First, let us examine this approach by taking a look at a fourth grade class taught by Mr. Kosho Masaki, a teacher at the Elementary School attached to the University of Tsukuba (Sansuka: mondai kaiketsu de sodatsu chikara, Toshobunka 1985).

1-1. Fourth Grade Class on Parallelism Taught by Kosho Masaki

To introduce parallelism, Masaki started by drawing a sample lattice pattern. The following process shows how children developed the idea of parallelism in his lesson study.



<u>Task 1. Let's draw the Sample 1 lattice pattern</u>

All of the children were able to draw this lattice pattern by taking points spaced evenly apart along the edges of the drawing paper and drawing lines between them. "It went well!"

<u>Task 2. Let's draw the Sample 2 lattice pattern</u>

The children began to draw the pattern based on a diagonal line moving upward to the right. What kind of reactions do the children have? The results are varied and depict several different strategies. However, they can generally be categorized into the ways shown in Drawings A and B.

<u>Developmental Discussion: "What?", What happened in Task 2?</u>

Masaki explained his problem solving approach as follows. In this (dialectic) situation, the children, even those who completed the task mechanically, were asked why they were able to draw the pattern in Task 1 but not the pattern in Task2. They were asked to try to find various ways in which to draw lines in order to reproduce the pattern shown in Samples 1 and 2. Because the children saw that others came up with results different from their own and everyone grew in confidence from their ability to draw the pattern, they began asking one another "How did you draw that?" and "Why did you think you could draw it by doing it that way?" They found it necessary to discuss their results. They began to distinguish between methods and to develop explanations. Through this developmental discussion, they were able to produce the word 'parallel' for what they had found based on what they had learned from others.

When children become aware of the unknown – in other words, there is a gap in their knowledge or meet different ideas – they become confused and think "something is wrong." This is then followed by a sort of conflict, leading to the questions "What?" and "Why?" Furthermore, when children enter developmental discussion (a dialectic) and are faced with ways of thinking that are unknown to them (knowledge gaps with others), it also causes conflict, forcing them again to ask "What?" and "Why?" Here again, they have to compare their way of thinking with that of others, evaluate it again by themselves and discuss their findings with other children. In this sequential flow, children make use of what they previously learned to turn the unknown into newly learned knowledge (a new understanding). This is the problem-solving approach discussed in this book based on conflict and understanding.

Here, one must ask why then did all the children feel that the drawing in Task 1 had "gone well," but in Task 2 two distinctly different types of drawing appeared. The reason lies in the diverse ways of thinking that appear in the sequence of tasks. In the next section of this chapter, we will clarify this using the terms 'conceptual or declarative knowledge' and 'procedure (form and way of drawing).' Then based on these terms, the sequence of tasks is analyzed again.

1-2. Looking at Masaki's Class in Terms of Meaning and Procedure

Meaning (in this instance, conceptual or declarative knowledge) refers to contents (definitions, properties, places, situations, contexts, reason or foundation) that can be (re)described as " ~ is ..." For example, 2 + 3 is the manipulation of '004—000'. The meaning can also be described as: "2 + 3 is 0 04—0 00" and as such explains conceptual or declarative knowledge. In Mr. Masaki's class, this method can be used to explain as follows: "The sample model is parallel lines." It therefore describes the meaning, which subsequently becomes the foundation of creating conceptual or declarative knowledge regarding the parallelism of the sample model.

Procedure (in this instance, procedural knowledge) on the other hand refers to the contents described as "if...., then do..." This is the procedure used for calculations such as mental arithmetic in which calculations are done sub-consciously. For example, "if it is 2×3, then write 6" or "if it is 2 + 3, then write the answer by calculating the problem as 0 04—0 00." This is procedural knowledge.

By doing this, you may say, "Oh, I see, the meaning is merely another expression of the procedure, that's why they match." Yes, that is true for those who understand that they do match. However, people do not immediately understand that they match. Even if they know that the sample models are graphs of parallel lines (conceptual knowledge), this does not mean that they can draw them (procedural knowledge). On the other hand, even if people can draw parallel lines (procedural knowledge), it does not mean that they understand the conceptual meaning (properties, etc) of parallelism. Cases when conceptual and procedural knowledge do not match are not only evident in mathematics classroom, but also in other facets of everyday life. For example, despite knowing their alcohol limit (conceptual knowledge), there are cases when people drink too much. Furthermore, it is this mismatch and contradiction that becomes the catalyst for the process in which people encounter a conflict, experience reflection, deepen their knowledge and gain understanding.

Let us return to Masaki's class. At first glance, the way of drawing pattern 1 in the first task appears to be a general method for drawing figures. However, from the and B in task 2, it seems that the children confused the two procedures shown in the box. Even if the children produce the same problem, how they acquire conceptual and procedural (form and way of drawing) knowledge, and the use of that understanding and knowledge are many

<u>Way of drawing 1: Procedure a</u>
→Way of Drawing A; Task 2
If you want to draw the model, draw lines spread
perspective of the ways shown in Drawing A evenly
apart from the top edge of the paper.
<u>Way of drawing 1: Procedure b</u>
→Way of Drawing B; Task 2
If you want to draw the model, draw lines spread
answer, the ways they understood the evenly apart.

and varied.

Based on analysis of the ways shown in drawings A and B, Masaki's class is described by conceptual and procedural knowledge.

The gap between the Sample model (conceptual knowledge) and the way of drawing (procedural knowledge): encounter a conflict

- Thinking "hold on, I can't draw this using procedure a; the lines cross over if extended, but as shown in the samples, the lines do not cross."
- "Why was I able to draw Sample 2 pattern using procedure b and not procedure a?"

Reviewing the way of drawing (procedure), and revising and reconsidering the semantic interpretation of the Sample model, which acts as the foundation of the drawing method.

- "How did you draw that? Why did you think it would work out if you did it that way?"
- Reason (coming from semantic interpretation of the Samples); lines in the Samples are
  all evenly spread apart, so they don't cross over.
- "I tried to draw the lines spread evenly apart, but they crossed over. How should I do it?"
- How do you properly draw lines spread evenly apart? By using the correct drawing
  method, which makes right angles and alternate interior angles evident.

Elimination (bridging) of the gap between the semantic meaning and way of drawing (procedure): to a coherent understanding

Taking into meaning (even spreading of lines, no crossing-over, and characteristics of right angles, corresponding angles and alternate interior angles), designation (definition) of parallel and drawing method (procedures including equal spread of lines, right angles, corresponding angles and alternate interior angles).

Within the developmental discussion process, procedure b, in which lines are drawn equidistantly at all points, works for both Samples 1 and 2. In contrast, procedure a, in which the lines are drawn from the top edge of the paper, clearly works for Sample 1, but does not work for Sample 2. Because Sample 1 is contrasted with Sample 2, the meaning of equal spread of lines is connected to the method of drawing with attention to lines equidistant at all points, right angles, corresponding angles and alternate interior angles. As a result, the basis (meaning) of why that way of drawing was attempted is explained by the children's comments.

Naturally, Masaki anticipated and expected to encounter undifferentiated schematic interpretations and drawing methods on the part of the children, and as such planned his classes accordingly. The teacher does not start by teaching the meaning and way of drawing parallel lines he is familiar with, but in fact starts by teaching at a level which assumes that children have not yet learned the word ‘parallel.’ The teacher tries to make use of previously learned methods of drawing parallel lines (procedures) that the children already know. By confirming previously learned knowledge, the teacher instills a sense of efficacy through leading children to a successful completion of the task. Following that, the teacher then makes the children face the difficulties by questioning “What?” at times when it does not work well. Due to the conflict that arises, children then ask about the meaning of the parallel lines. The teacher aims to have the children create their own reconstruction of the method of drawing and the meaning, using what they already know as a foundation.

Looking back, it can be seen that the flowchart presented on the right is embedded in Masaki’s class. As is indicated, the class is structured in such a way that the children proceed from a feeling that everything is “going well” to suddenly asking “What?”. This transition serves as the context in which a diverse range of ideas appears regarding how the children have understood the problem and what type of meanings and procedures they have acquired. This class is indeed a type which solves problems through developmental discussion (a dialectic) and makes use of a diverse range of ideas by overcoming the conflict of “What?” sorting through and clearing up previously misaligned meanings and procedures, and finally reaching a stage of understanding.

Dialectic Structure of Mr. Kosho Masaki’s Parallel

Confirming Previously Learned Knowledge
Situation: Task 1 “It goes well”—S**ense of Efficacy**
Even if gaps in meaning and procedures exist, they do not appear here.
Different Situation from Previously Learned Knowledge: Task 2
There are children who show gaps in their understanding of meaning and procedure and some who don’t.
“What?” – **Conflict**
Developmental discussion (a dialectic) by questioning new meanings and procedures
Acquisition of a **Sense of Achievement** by Overcoming the Conflict and Proceeding **through Understanding**

2. Reading the children’s diverse range of ideas through meaning and procedure (form and way of drawing)

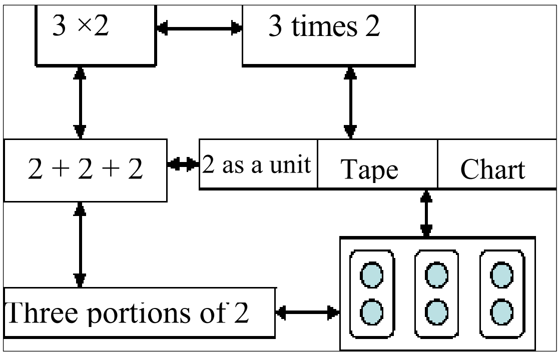
For the planning of a lesson on the Problem Solving Approach, it is necessary to anticipate the diversity of children’s responses and plan a developmental discussion for studying the target of the lesson. This section shows ways of reading and anticipating children’s ideas using the words ‘meaning’ and ‘procedure (form and way of drawing).’ The theory of conceptual and procedural knowledge in mathematics education by James Hilbert (1986) is well known, and in Japan, Katsuhiko Shimizu applied a similar idea in classroom research (1986). Meaning and procedure for lesson planning theory has been developed by Isoda (1991) as an adaptation of cognitive theories to the progressive development of mathematics ideas within lessons.

To begin with, we would like the readers to read once more the above-mentioned explanation of meaning and procedure, and do the following exercise.

> **Exercise 1**
> Which do the following correspond to: meaning or procedure?
> 1. Reduction to the common denominator refers to finding the common denominator without changing the size of the fractional number.
> 2. In order to compare the size of fractional numbers, either reduce or increase the fractional number size.
> 3. In order to divide by a fractional number, take the reciprocal of the divisor and multiply.

2-1. What is meaning? What is procedure (form and way of drawing)?

a. What is meaning?



Meaning (conceptual knowledge) can be illustrated by "man is a wolf," for example. Of course, a man is a human being, but by likening man to a wolf and changing the way of saying it, one can make a sentence that aims to express the meaning of "man." The previous example "2+3 =004-000" gives a concrete example and changes the way it is said to express the meaning. The mathematical expression "3X2 = 2 + 2 + 2" also expresses meaning (in Japanese, 2X3 means 2 + 2 + 2). It is a rephrasing too. Such a rephrasing not only refers to a concrete example but also refers to what is already known. Note that the meaning of multiplication that children learn in the second grade can be summed up as shown in the figure above. The characteristics of the meaning are seen in the fact that a number of elements are connected like a net, and as such, we as teachers think that children can understand the meaning in more diverse ways when we are able to interpret like this. The important thing regarding diverse expression is that the meaning is in fact picked out and expressed through such rephrasing.

In response to the problem "How many L and dL is 1.5 L?", a student replied: "Before, we learned that 1 L is 10 dL, and that 1 dL is 0.1 L. If I use that, 1.5 L is 15 parts 0.1 L. 10 parts 0.1 L is 1 L. The remaining 5 parts are 5 dL. So, 1.5 L is 1 L and 5 dL." When that child explained the basis of her reasoning, we as teachers can see that the child has made a deduction and explained it based on meaning.

b. What is procedure?

Procedure (procedural knowledge, form, way of drawing, method, pattern, algorithm, calculation, etc.) can be expressed as follows: "If the problem is to divide by a fractional number (recognizing conditional situations), then take the reciprocal of the divisor and multiply." The first characteristic of procedure is being able to process automatically, without question, and instantly. However, proficiency (in other words, practice) is necessary. When answering the question how many dL are in 1.5 L, take a case where a student rapidly answers “1 L 5 dL.” If the student automatically follows the rule “if L is interpreted as L and dL, then focus on the position of the decimal point and think of L as coming before it and dL as coming after it,” then one could acknowledge that this student is using procedure. Being able to solve a problem instantly like this by using procedure means that we have come to a stage where we can find a solution without having to spend a lot of time deducing meaning, which in turn brings us to the point where we can consider reducing thinking time (e.g. short-term and working memory). Another characteristic of procedure is that it produces new procedures such as the complex grouping of the four operations, as seen in the example of division using vertical notation (long division) whereby numbers are composed (estimating quotient), multiplied, subtracted and brought down (to next lower digit). If each procedure is not acquired, it is difficult to use complex procedures that incorporate some or all of them. In other words, if one becomes proficient, it does not matter how complex the grouping of procedures are, as one will be able to instantly use them. Simplifying complex deductions and being able to reason about a complex task quickly means that one is able to think about what else should be considered.

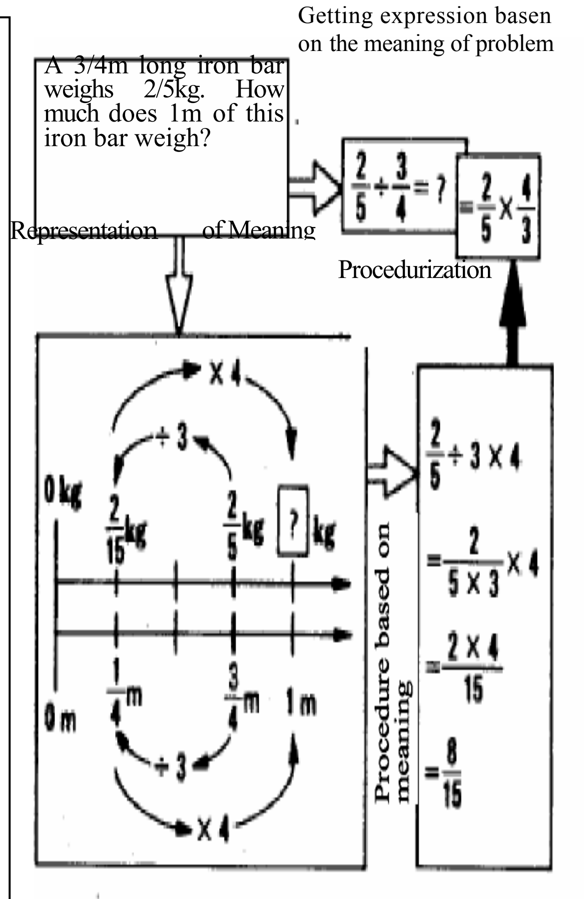
c. The relationship between meaning and procedure

As was shown in the method of drawing and the meanings of the patterns in Masaki’s class, there are instances when the meaning and procedure match (no appearance of gaps, consistency of use) and other instances when they do not match (appearance of gaps, inconsistency). In learning process through the curriculum or planned sequence, there are situations where the meaning and the procedure contradict each other and situations where they do not. Moreover, from the curriculum/teaching-learning sequence perspective, these two instances are mutually linked or translated as follows.

<u>Procedures can be created based on meaning</u> (**the procedurization of meaning**, in other words, procedurization from concept). For example, when tackling the problem “How many L and dL is 1.5 L?” for the first time, a long process of interpreting the meaning is applied and the solution “1.5 L is 1 L 5 dL” is found. Additionally, this can be applied to other problems such as “How many L and dL is 3.2 L?” with the answer being “3.2 L is 3 L 2 dL.” Children soon discover easier procedures by themselves. Simultaneously, children realize and appreciate the value of acquiring procedures that reduce long sequential reasoning to one routine, which does not require reasoning.

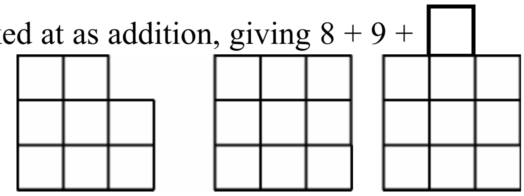
There is a remarkable way to shorten the procedure from known concept and procedure. The example, “if the problem is the division of fractional numbers, then take the reciprocal of the divisor and multiply” is shown in the diagram below. Using the previously learned concept of proportional number lines, the meaning of the calculation is represented and the answer is produced based on this representation. As a result of this representation, the alternative way of calculation ‘take the reciprocal of the divisor and multiply’ is reinterpreted so that it can be produced simply and quickly from an expression of division. Thus children reconstruct a procedure that can be carried out simply and quickly by reconsidering the result based on meaning. Even in a simple case such as the multiplication 2 times 3, this is $3 + 3 = 6$ as a meaning, but as a procedure, 3×2 is interchangeable with the memorized result of 6. This remarkable way is also the procedurization of meaning. Many teachers believe that the procedure should be explained based on meaning, but the alternative is often preferred because it is much simple and easier. Using one of the key values of mathematics, namely simplicity, we finally develop procedure based on meaning.

Meaning becomes the foundation for acquiring procedure. When children struggle to use previously learned knowledge, and if they employ a d versity of meanings for producing procedure, the importance of faster and easier procedures for obtaining answers will become clearer, as the alternative is to follow the difficult path of long reasoning. By debating diverse meanings in order to reason, children can clarify meaning and thus may recognize the situations for which the produced procedure is applicable. Procedure has the ‘if, then’ structure. The ‘if’ describes the conditions of applicable situations; when applicable, it is acceptable to carry out the ‘then’. Negotiating meaning is important for understanding applicable situations, even if it is very difficult to clarify the conditions for applicability without extension (the notion of extension is explained later).



The above is an example of how procedures can be created based on meanings. However, the reverse can also be achieved: meaning can be created based on procedure (meaning entailed by procedure, in other words, conceptualization of procedure). Let us consider this notion from the perspective of addition taught in the first grade and multiplication taught in the second grade of school. In the first grade, as in the operation activity where ‘○○ ←○○’ means 3 + 2, children learn the meaning of addition from concise operations and then become proficient at mental arithmetic procedures (the procedurization of meaning). At that point, calculations such as 4 + 2 + 3 and 2 + 2 + 2 are done more quickly than counting, which is seen as a procedure. Further, in the second grade, compared with the case of several additions, only repeated addition problems lead to the meaning of multiplication. It is here that the specific procedure known as ‘repeated addition’ (now called multiplication) is incorporated into the meaning (meaning entailed by procedure). The reason such situations are possible is that children become both proficient at calculations using addition and familiar enough with the procedure to do it instantly. Children also see the meaning of

a situation such as in the following picture showing three groups of objects. To find the total number of objects, it can be looked at as addition, giving 8 + 9 + 10, but by moving an object from the third group to the first, it can be looked at as repeated addition or multiplication, giving 3×9. Children unfamiliar with the procedure resort to learning addition and multiplication at the same time, which in turn make it more difficult for them to recognize that multiplication can be regarded as a special case of addition.



Only people who have a good understanding of the meaning and the procedure use them as if they were one; they can be thought of as two sides of a coin, each of which has different features but together form the one coin[1]. On the other hands, from the curriculum sequence and its teaching-learning perspective, meaning and procedure develops mutually. Due to the fact that meaning can become procedure and vice versa in the teaching-learning process on the curriculum, only teacher can recognize the situation which meanings and procedures do not related mutually and plan how to develop mutual relationship. As this book aims to support teachers in their lesson planning, it is up to each teacher to decide what is meaning and what is procedure in each class in accord with the actual situation of the children and the classroom objectives.

2-2. Using meaning and procedure (form and way of drawing) to anticipate children's ideas

In the problem solving approach, teachers anticipate children's ideas in order to plan to develop their ideas using what is already known. Meaning and procedure support this anticipation[2].

a. Knowing meaning and procedure allows you to anticipate children's incomplete ideas

Some months after learning how to divide fractional numbers, children are asked: "Why does that happen?" Many children reply "because you turn it upside down and multiply" (procedure), even though they could answer with meaning when they first learned well about it. This indicates that they lose meaning in exchange for procedural proficiency (proceduralization of meaning). Here we would like readers to answer Exercise 2, keeping in mind children who tend to forget the meaning.

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[1] The metaphor is as same as Sfard but the idea itself developed originally at the publication in 1991 as a result of lesson study with elementary school teachers. She had pointed the same idea.

[2] In Japan, curriculum standards are fixed and textbooks are distributed by the government. One of the basic curriculum sequence and textbook contents sequence in Japan is 'extension' or 'expansion', that is, extending learned procedures to new situations. Depending on the situation, teachers can share children's responses through the Lesson Study and teachers' guidebooks, and at the same time, they can anticipate children's reasoning and the process of discussion.

Exercise 2 Expressing a number with one denomination in a form that uses multiple denominations

A third grader with previously learned knowledge able to quickly give the answer “1.5 L = 1 L + 5 dL” is asked the following question: “4.2 m = how many m and how many cm?” Anticipate the child’s reaction.

A procedure that a child becomes proficient in is typically swimming or riding a bicycle; it is not easily forgotten, but meaning does not stay in one’s consciousness unless it needs to be used. The most common answer by children to the above exercise, as expected, is “4.2 m = 4 m + 2 cm.” In the third grade, children are taught to work as far as the first decimal point in small numbers. Therefore, when learning, children are usually only faced with units of 1/10 such as in L and dL, or cm and mm. Children who become able to quickly give the answer “1.5 L = 1 L + 5 dL” only experience the situation where that procedure is applicable. As a result, they become unable to make semantic judgments on when that procedure can be used.

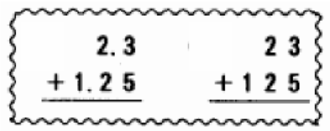
The correct procedure “If ..., then...” will always produce the correct result as long as the conditional “if” part of the semantic judgment is correct. However, when children only experience applicable instances, they over-generalize the meaning and become unable to make a correct judgment. As a consequence, many children who use this so-called ‘quick/instant’ procedure may use it in cases where it does not apply.

It should be noted that this quick response procedure is not only something that the teacher has taught, but rather is an extremely convenient idea that the children may have arrived at on their own. Even if this concept is invalid, children will not recognize this as long as they continue to be presented with tasks that do not show the weaknesses of the invalid concept. For example, even if children from Mr. Masaki’s class completed the first task using an invalid concept, the underdeveloped nature of the concept would not become apparent until it was applied to another task. Therefore, what the teacher should first recognize is a child’s idea created as his/her own. From there, the next step is to deepen that idea by investigating whether or not that idea can be generalized to other tasks. This is the challenge for teachers.

b. Gaps between meaning and procedures appear in extending situations

As presented at the beginning of this chapter, the steps "It goes well! It goes well! What?" are important. As long as everything goes well and is applicable in the end, the gaps between meaning and procedure will not become a problem. In such a situation, children are not faced with a difficult situation; they are within the range of previously learned knowledge, and have not yet been challenged by the unknown. However, a situation when something does not go well or when there is a need to close a knowledge gap is indeed one where true discoveries and creations are made. When a person thinks "What?" in a situation, this indicates issues that should be given genuine thought. An example of when things do not go well is the 'extending situation.'' In an extending situation[3], the gap between meaning and procedure appears as diverse ideas. Here, let us look at the example of the extension of a procedure from whole numbers to decimal numbers.

Example 1) shown on the right is an over-generalized idea that can be seen in the decimal number calculation. It is usually explained as misunderstanding the meaning of place-value.



Why does this type of idea appear? It arises because, when calculating with whole numbers using vertical notation as in example 2), the proper procedure is to write the numbers so that they are aligned on the right side. Example 1) indicates that the whole number procedure that was previously learned was applied. Having only experienced the calculation with whole numbers, the child is aware only of the procedure of aligning numbers on the right. Furthermore, the child has learned the procedure of right alignment through his or her experience of learning whole number calculation using vertical notation.

The diagram on the next page illustrates the process of the extension of the application of the whole number procedure. With regard to the introduction of whole numbers in situation I, the procedure for aligning decimals matches the meaning of place-value (arrow A). When children become accustomed to this procedure, they forget the meaning of place-value and become proficient in quickly aligning to the right (II). In the domain of whole numbers, the meaning of place-value is not contradicted even if numbers are aligned to the right (arrow B). However, when children apply this procedure to decimal numbers (III), it contradicts the meaning of place-value as shown in 1) (arrow C). Therefore, when children are faced with an instance when the procedure does not apply, they become aware of the gap and must once again return to the meaning of place-value. Then, they apply the procedure to both whole numbers and decimal numbers, and they become aware of the procedure of aligning decimal numbers as a procedure in accordance with the meaning of place-value.

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[3] Extension (extending or expanding situation) is a basic principle of Japanese curriculum and textbook sequence in mathematics. Thus, over-generalization by students can be anticipated by the teacher. The examples here may not be particularly special even for those in other countries because the extension is normal sequence in school mathematics without axiomatic mathematics at the age of

New Math.

Situation	Meaning Procedure	Explanation	Appropriateness
I Introduction of calculation in vertical notation using whole numbers	**Decimal notation System meaning** (A) **Write 2 3 + 5**	The meaning of a decimal notation system is based on the procedure of keeping decimal points in alignment. (The meaning and procedure match)	**Appropriate**
II Becoming proficient in whole numbers	**Align to the right and write** ***? (Forgotten)***	When children become proficient, they no longer need to think about the reason they follow that procedure. As a result, the procedure is simplified from the alignment of the decimal points to one of right-side alignment.	**Valid**
III Application of decimal numbers	**(No meaning)** **Align to the right and write**	The procedure for whole numbers is generalized for decimal numbers	**Inappropriate**

Obviously, many children solve decimal number calculations using vertical notation through an understanding of the meaning of place-value. Thus the number of children who resort to the right-side alignment procedure is small. From the perspective of meaning and procedure, however, the way in which gaps in meaning and procedure occur tells us that there is a necessity in the teaching process to separate meaning and procedure into the following three categories. Children's levels of comprehension are by no means uniform in the process of learning. Comprehension develops differently in each child. While there are children who are no longer aware of meaning because they have become accustomed to applying quick and easy-to-use procedures, there are also children who are aware of meaning and use it as a basis for the procedures. Because the conditions vary, a diverse range of ideas involving previously learned knowledge appears in situations (extending situations) (III) when easy-to-use procedures do not work.

I) Deepening meaning: No appearance of gaps between meaning and procedure
"It goes well!"
II) Gaining an easy-to-use procedure from the meaning: Gaps are unrecognizable.
"It goes well!!"
Children become accustomed to easy-to-use procedures that work and many of them become unable to recall the meaning.
III) Situation where easy-to-use procedures do not work: Awareness of gaps
"What?"

The problems considered in Mr. Masaki's class and in exercise 2, the practice of expressing a number with one denomination in a form that uses multiple denominations, are examples of extending situations (expansion). In an extending situation, the procedures and meanings that have been established will not work, which means that they will need to be reconstructed. Taking the above decimal number calculation in vertical notation as an example, the meaning of place-value works, but the right-side alignment procedure needs to be revised. Accordingly, the meaning of place-value needs to be reviewed, and the procedures used need to be revised to ones that align the positioning of the decimals in accordance with proper place-value notation. In short, as an educational guide, category III can also be described as follows:

III') Reviewing of meaning and revision of procedure: Elimination of gaps

2-3. Diverse ideas can be classified by meaning and procedure

Up to this point, we have focused on the most extreme over-generalized ideas (misconceptions) to indicate the occurrence and elimination (bridging) of gaps between meanings and procedures. Naturally, in actual classes a diverse range of ideas will surface, including correct and wrong answers. In order to plan developmental discussions, it is necessary to anticipate the type of diverse ideas that will most likely appear. Here, let us treat the children's ideas as observations. For example, at the Sapporo City Public Konan Elementary School, Hideaki Suzuki's 5th grade class looks at division involving numbers with 0 in the end places. This class, as was the case with Masaki's class, first confirms previously learned knowledge of division when there is no remainder (task 1) and then moves on to the target content, which has yet to be learned: division when there is a remainder (task 2). The objectives of this class can be confirmed in the following discussion showing the flow of the class lesson (See next page).

Task 1. Known problem to confirm a previously learned procedure and the meaning it is based on: Previously learned task.

When children who have knowledge of basic division work out the answer to 1600÷400, the following is reviewed:

$$\begin{array}{r} 4 \\ 400\overline{)1600} \\ \underline{1600} \\ 0 \end{array}$$

A. Take away 00 and calculate: procedure
B. Explain A as a unit of 100 (bundle): meaning
C. Substitute A for a 100 yen coin and explain: meaning

Task 2. Unknown problem that seeks an application or expansion of the previously learned meaning and procedure: Target task.

The target problem presented is 1900÷400, which presents a problem for some children and not for others as to how to deal with the remainder. As a result, the following ideas appear.

a) Answer to the question using a procedure in which the meaning is lost.
Apply A and make the remainder 3. Because the meaning is lost, the children do not question the remainder of 3. (Half of the class)

b) Answer to the question when procedures have ambiguous meanings.
Using A and B, the remainder was revised to 300. However, because the meaning was ambiguous, it was changed to 400. (Several students)

c) Answer to the question when the procedure is ambiguous.
A was used, but here a different procedure was selected by mistake. No students question the quotient 400. (Very few students)

d) Answer to the question that confirms procedural meanings.
Using A, an explanation of the quotient and remainders from the meaning of B and C.

a)
$$\begin{array}{r} 4 \\ 4\cancel{0}\cancel{0}\overline{)19\cancel{0}\cancel{0}} \\ \underline{16} \\ 3 \end{array}$$

b)
$$\begin{array}{r} 400 \\ 4\cancel{0}\cancel{0}\overline{)19\cancel{0}\cancel{0}} \\ \underline{16} \\ 300 \end{array}$$

c)
$$\begin{array}{r} 400 \\ 4\cancel{0}\cancel{0}\overline{)19\cancel{0}\cancel{0}} \\ \underline{16} \\ 3 \end{array}$$

d)
$$\begin{array}{r} 4 \\ 4\cancel{0}\cancel{0}\overline{)19\cancel{0}\cancel{0}} \\ \underline{16} \\ 300 \end{array}$$

Why do answers differ?

Where did you get lost? What did you have a problem with? A reminder of conflict through solving an exercise using your own ability.

By reviewing the solution process, the basic meaning is reconfirmed and the procedure for dealing with remainders is learned.

First, the children grapple with Task 1, which they have learned before. The teacher links this task directly to Task 2 in the target content of the class, keeping the children's solutions in mind. This is done by asking the children to confirm the procedure for the division using vertical notation, and asks them why it is not a problem to do this (meaning). Simultaneously, the teacher makes sure the children are able to explain both procedure and meaning. Following that, the children tackle target Task 2, which requires them to deal with remainders. In Task 2, a variety of ideas (a-d) appear among children who are doing the work without complete knowledge of the meaning, and among children who are confirming the meaning while working on the task.

Situation: confirming what they have already learned "It goes well": **Sense of Efficacy**

Mutual confirmation of meaning and procedure. Even if gaps in meaning and procedure exist, they do not appear here.

Situation: different from what they have learned before—Conflict

What?: **the unknown** due to an awareness of the gap with what they have already learned.

Some students experience such gaps in meaning and procedure whilst some do not.

What?: **Surprise at the difference in ideas** with other students and reflection on one's own ideas. Developmental discussion that correctly redefines meaning and procedure.

Acquisition of a sense of achievement, appreciation, by overcoming conflict and proceeding through to understanding

The objective this time is to have a developmental discussion regarding the place-value of the remainder being adjusted to the place-value of the dividend.

Here, it is important to have readers understand that the above approach is fixed in the class. It is worthwhile noting that even if meanings and procedures are previously confirmed, there is a diverse range of ways to process and implement that comprehension. As such, a variety of ideas appear. The starting point in the creation of diverse ideas lies in ways to process and utilize individually.

When categorizing the variety of ideas above (a-d) by meaning and procedure, the following category types can be identified. These are developed with reference to the extension task that followed the known problem used to confirm previously learned knowledge.

Type 1. Solutions reached through the use of procedures without (or regardless of) meaning: Prioritize procedure without meaning.

This is the above-mentioned idea a). It refers to an idea reached through consideration without much attention to meaning, even though the correct procedure (calculation) is applied. There are children who immediately change their ideas by recalling the meaning after having been asked to explain or after listening to other children's ideas. However, most children substitute meaning with procedure and when they are asked for an explanation they usually reply by describing their procedure, saying "I did this, then I did that." Prioritizing the procedure means that the children do not give careful consideration to the meaning; rather they tend to use quick procedures.

(In the case of an already known task, and if we apply the correct procedure, the answer must be appropriate, but now we are discussing the case of the extension task.)

Type 2. Solution reached through the use of procedures with meaning: Prioritize procedure with confused or ambiguous meaning.

This type is composed of ideas b) and c). These students have the intention of confirming the meaning of the calculation procedure, but their idea includes their own semantic interpretation. Therefore, when getting to the core of their idea, it is found that their idea is one that contradicts the meaning and procedure they have previously learned. As a result, there are many instances in which their idea brings about confusion and unease.

Type 3. Solution reached through the use of procedures backed by meaning: Secure procedure and meaning.

As shown in d), when a solution reflects the appropriate meaning and has been learned as a procedure, there are no contradictions between procedure and meaning.

Usually, when people are faced with a task they are unfamiliar with, the first thing they do is to test existing quick-to-use procedures in which they are proficient. This is what is referred to as the 'prioritize procedure' situation. If children believe in the situation that they got appropriate answere without considering meaning, then they are categorized as Type 1: 'prioritize procedure without (or regardless of) meaning.' In actual fact, there are many children who react to an unfamiliar task by prioritizing procedure without giving any careful thought to meaning. If children further investigate meaning when asked if the procedure they chose to implement is appropriate, and they show confusion and concern, they are categorized as Type 2: 'prioritize procedure with confused or ambiguous meaning.' In contrast, a careful student who tackles a problem by always investigating the meaning and making sure there are no gaps will produce a result that has a secure procedure and meaning; they are categorized as Type 3.

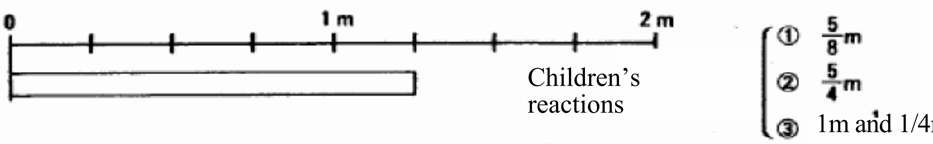
Although not shown in the above example, other ideas such as the following are also identified.

Type 4. Solutions through meaning only: Prioritize meaning without procedure (or confused procedure).
This type is seen when the procedure cannot be used appropriately or the student is not yet proficient in its use. Consequently, the solution is gained through thinking mainly about the meaning. As an example, consider the case where a student cannot calculate 1900 ÷ 400, but can answer if asked to solve the problem: “You have 1900 yen. You buy as many 400 yen pencil cases as you can. So...”
Type 5. Inability to find a solution due to insufficient meaning and procedure: No meaning or procedure.

It is particularly important for teachers to keep in mind Type 5 children who are unable to solve a problem. In the case of Type 4 children, they can give many possible reactions in class, but in many cases there is no result when it comes to formal tests. In the case of 1st and 2nd graders, many Type 4 children give reasonable answers if they have a good understanding of the meaning, but children in higher grades will meet difficulties. When elementary school children reach the 5th and 6th grades, and even more so when they enter junior high school, there is an increase in textbook and course materials that require the procedurization of meaning. If children do not have procedure, it is impossible to develop the meaning entailed by the procedure. So it is very important to be aware that some children in Type 4 will move into Type 5 without proficiency of procedure.

Here we would like readers to tackle the following problem regarding the meaning and procedural knowledge possessed by children from Katsuro Tejima’s class.

Exercise 3 The following is used in the introduction of fractional numbers for 4th graders. When asked to answer using fractional numbers for the length of a piece of tape, children’s responses fall into one of three different types. Please explain what the children were thinking.



Answers to Exercise 3

As previously taught in the third grade, a fraction is interpreted as a number of parts of the equal (even) divisions of a whole, and in the case of the fraction of a quantity, “2/3 m is the same as two parts of three equal divisions of 1 m”. Fractions of one meter are learned only in the context of measurements of less than 1 m. This previously learned procedure tells children always to divide the whole number evenly and uses contexts in which the numerator never exceeds the denominator. The children’s thinking may then be characterized as follows.

1. 5/8 m: the procedure was applied by making 2 m as one unit. This method is consistent with the procedure already learned; however these children did not recognize the contradiction inherent in obtaining a value less than 1. Accordingly, it illustrates Type 1: ‘prioritize procedure without meaning.’
2. 5/4 m: this answer was quickly found using the assumption that if there were three parts, each of which was 1/4 m, the total length would be 3/4 m, so that if there are five parts, the length should be 5/4 m (generalization of procedure). This contradicts the meaning and procedure children were previously taught, in which a numerator is smaller than denominator. Children who felt uneasy in this instance would be classified as Type 2: ‘prioritize procedure with confused meaning.’ Children who used the diagram to establish that 3 parts of 1/4 m becomes 3/4 m and so 5 parts becomes 5/4 m (meaning), but were then confused as to whether they could write that way because they had previously learned that the numerator cannot exceed the denominator (procedure), would be classified as Type 4: ‘prioritize meaning without procedure (or confused procedure).’
3. This answer shows that the children regarded the length as 1 m together with a further 1/4 m, obtained by subtracting 1 m from the total. As there is no discrepancy with what was previously learned, these children would be classified as Type 3: ‘secure procedure and meaning.’

This book focuses on lesson planning by teachers, and as previously mentioned, teachers ought to decide what the meaning and procedure are in their class material, and should provide appropriate educational guidance in accord with their teaching plan. It should be noted however that even when children are classified as the same type, their actual understanding, their thought processes and the ways they deduce meaning and procedure, may differ depending on the individual child and the situation.

Before each lesson, it is necessary for teachers to prepare teaching material and plan the lesson on the basis of the required curriculum sequence. In aiming to support lesson planning, this book has identified the above-mentioned types as part of the teaching material research carried out by the teacher. The teacher will be able to prepare the following in accord with the categorization by types: anticipate what kind of ideas will emerge from children based on what they have previously learned; plan well-devised instructional content for the class based on these diverse ideas; and create ways of facilitating the instruction so that children are able to recognize what they do not understand and are then led to experience the joy of understanding. By anticipating children’s ideas and the causes of possible confusion, teachers will be able to envisage beforehand how they should develop their explanations and discussions. The categorized types provided are for the teacher to use in order to plan lessons for conceptual development, based on what the children have previously learned, using extending examples or situations.

3. Planning for a Lesson with Developmental Discussion and Diverse Ideas

This section will incorporate what has been covered in previous sections and will demonstrate how to implement the wide range of ideas children create and show how to run a developmental discussion (dialectic) in the lesson. As already mentioned, the developmental discussion is planned for special occasions during the teaching sequence. If the curriculum or textbook sequence includes extending mathematical ideas, we can expect contradictions to inevitably occur. In the problem solving approach, we aim to develop mathematical communication as well as mathematical conceptual development. Thus, in this book, we are quite positive in promoting such contradictions as objects for discussion in the mathematics classroom.

3-1. Instruction planning in which a wide range of ideas appears by taking advantage of knowledge gaps

Here, the 'third grade decimals' lesson conducted by Junko Furumoto (Sapporo Midorigaoka Elementary School) will be used as an example. When teaching fourth grade lessons on decimals, it is known that children tend to over-generalize when they try to express a number with one denomination in a form that uses multiple denominations, as shown previously in Exercise 2. Ms. Furumoto recognizes this over-generalization as a gap that appears due to an extension of the procedure children have developed for dealing with numbers with only one decimal place to numbers with two decimal places. Accordingly, she has created the following lesson plan to take advantage of this gap and so add depth to her lesson on decimals.

1st class: In what situations are decimal numbers used? The existence of decimal numbers. **2nd class:** How much juice is there? The need for decimal numbers (meaning). 1/10 dL = 0.1 dL: decimal numbers are used to express amounts smaller than one unit (meaning) **3rd class:** Let's make a numeric line based on 0.1: the size of decimal numbers	*It goes well!* **The meaning and procedure match.**
4th class: Let's get decimal numbers to introduce themselves: practice with large/small numbers and amount (meaning and procedure). "I am 2.8. I am a number made up of two 1s and eight 0.1s." **5th class:** How much is 3.7 cm or 1.5 L: practice in use of single and multiple denomination numbers. Re-expressing single and multiple denomination numbers (meaning and procedure).	*It goes well!!* **Procedurization, loss of meaning, or no loss of meaning.**
6th class: There are two pieces of string: one is 4.2 m and the other is 4 m 10 cm. Which one is longer?	*What?* **The occurrence of gaps.**

The first five lessons, each of which is one hour long, are designed to deepen the children's understanding of the meaning of the first decimal place. In particular, the fourth and fifth hours focus on procedural proficiency (form) in terms of semantic interpretation. Up to this point, the method of instruction is standard. The sixth lesson is planned to make children wonder "What?" A diverse range of ideas appears as some children try to apply quick, easy-to-use procedures while others consider the problem using their understanding of decimal numbers, based on the example that 0.1 equals 1/10 of 1. It is planned this way so that conflict will occur. Furthermore, this conflict is used to get children to re-evaluate the meaning of place-value, including those children that did not have an accurate understanding of the meaning of decimals in the first instance.

The sixth class unfolds as follows:

- Preconception: It's 4.2 m! It's 4 m 10 cm!

How should I compare them?

- The units are different, so if I don't align them, I won't be able to compare them.

What should you do so that you can clearly find out which

- For children who can't solve this problem by themselves, the teacher makes them realize

that they should use diagrams or the numeric value line they have previously learned.

a) 4.2 m = 4 m 20 cm, so... b) 4.2 m = 4 m 2 cm, so... (majority of the students)

c) 4 m 10 cm = 4.1 m, so... d) 4 m 10 cm = 4.10 m so...

- **Conflict:** a) vs. b), c) vs. d). Is 0.1 m 10 cm or 1 cm?
- **Returning to the meaning:** By converting the units to meters

(Using diagrams and number lines) 10 cm is 1/10 of 1 m, so it is 0.1 m

4 m 10 cm = 4.1 m < 4.2 m

By converting the units to cm

0.1 M times 10 equals 1 m, so it is 10 cm

4.2 m = 420 cm > 410 cm = 4 m 10 cm

If the units are different, then compare them by converting them (procedure)

- Reaching understanding

A wide range of ideas appear in answers a to d. Children chose answers a and c based on the meanings they had learned up to the fifth class: "0.1 m is 1/10 of 1 m" (classify as Type 3: 'secure procedure and meaning'). Answer b may be the result of the quick procedure in the fifth lesson, which doesn't work (classify as Type 1: 'prioritize procedure without meaning'). Answer d may be an example of Type 2: 'prioritize procedure with confused or ambiguous meaning' if the children are confused as to why a contradictory expression that they do not understand appears. This is due to the fact that should they consider the quick procedure 4.2 m = 4 m + 2 cm to be correct and actually write 4.2 m = 4 m 2 cm, they will also necessarily write

4 m + 10 cm for 4.10 m. A similar case is where children wrote 4.10, because 1/10 of 1 m is 10 cm. If the children are confused as to whether they can write 0 in the second decimal position, then they should be classified as Type 4: 'prioritize meaning without procedure (or confused procedure).'

After the gap in ideas has been confirmed[4], the class moves on to encouraging children who chose answer d (with a question about 4 m 10 cm being 4.10 m if 4.2 m = 4 m 2 cm), to consider the problem in the context of answer b, in order to return to the meaning of decimals they had previously learned, which is that 0.1 = 1/10 of 1. Through discussion, the quick procedure is revised and the procedure for converting the units becomes clear. Further, children's understanding of the meaning of decimals, which observes a place-value of numbers, such as 10 cm = 0.1 m, is deepened.

It is worth noting that even though the first five hours of lessons have placed heavy emphasis on amounts and meaning through the use of specific examples and number lines, a large number of children will choose answer b. As previously mentioned, when adults learn a quick procedure, they will try to use that procedure in the first instance. Children are no different. When children become aware of easy-to-use procedures, many children are unable to recognize the semantics of the pre-requisite 'if...' of the procedure (in the 'if..., then...' structure). Ms. Furumoto's children would not have acquired even the easy-to-use procedures sufficiently without attending the sixth class. Accordingly, the aim of the sixth class is to deepen children's knowledge regarding procedures that convert units and the meaning of place-value in decimal numbers by continuing to detect insufficient understanding and then revising the meaning.

The diagram below shows a summary of the sub-unit construction mentioned above, focusing on meaning and procedure.

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[4] The difficulty in understanding other's ideas is that each of them is deduced from reasoning based on the different presuppositions depending on different understanding. In order to understand each other, it is necessary to reason based on others' presuppositions or to identify the necessary presuppositions from which may be deduced other's ideas. This point is focused on the third book (Isoda&Kishimoto 2005).

I) Constructing meanings

1st – 5th class: Matching meaning and procedure. No gaps become apparent.

Specific amounts, number lines and diagrams are used to learn that 10x0. 1 amounts to 1 (meaning).

II) Constructing easy-to-use procedures with meanings as the base

Part of the 4th class: the following quick rewording is taught, “2.3 is made up of two 1s, and three 0.1s.”

Part of the 5th class: Becoming proficient in procedure. Some students begin to lose the meaning of the procedure.

5.3 cm = 5 cm 3 mm, 2.7 L = 2L 7dL can be re-expressed quickly.

III) The situation of easy-to-use procedures not working: Extending the situation The meaning is reviewed and the procedure is revised

6th class: the gap is exposed between the solution brought about from the procedure whose meaning has been lost and the solution that reflects the meaning. Then conflict occurs, leading to a review of the meaning of the procedure and a revision of the procedure itself. Through this, a new understanding is achieved.

Children who apply the procedure from the 5th class.
4.2 m = 4 m 2 cm
Meaning loss from the 1st to the 5th classes.

Children who solve the problem using the meaning learned from the 1st to the 5th classes.
4.2 m = 4 m 20 cm
4 m 10 cm = 4.1 m

The meaning of a place-value in decimal numbers is reviewed and acknowledged. Then the procedure for re-expression of numbers in different denominations is revised.

* The discussion structure of section III includes the Hegelian meaning of the dialectic process through sublation. Here, Other’s different ideas are functioning antithesis. We will discuss this later.

The climax of the sub-unit construction is section III. What is the process of reaching section III? First, in section I, procedures are learned while keeping meaning in mind. In section II, an easy-to-use procedure is acquired. As children become proficient in this procedure, some of them lose the need to consider meaning. In section III, they are faced with instances in which the easy-to-use procedure does not work.

At the stage of solving problems by themselves before whole classroom discussion, each child may become confused because the easy-to-use procedures do not always work. When they participate in developmental discussions, conflict arises regarding the difference in ideas held by other children. By experiencing that conflict, the meaning as a basis for supporting the procedure, which many children lost in section II, is once again recognized with a higher form of generality, and then the procedure is revised.

The following describes the process of sub-unit construction in more general terms.

The development of sub-units in which divers ideas appear

I) Deepening the meaning: when meaning and procedure match

Here, meaning is deepened by being matched to procedure.

It goes well!

II) Constructing easy-to-use procedures based on meaning

Here, the 'procedurization' of meaning is developed and students become proficient in easy-to-use procedures. At that time, some students fail to remember the original meaning. Even in such cases, however, the procedure will continue to result in a correct answer and no gaps in understanding will become apparent. Therefore, students experience no confusion.

It goes well!

III) A situation when easy-to-use procedures do not work; a review of meaning and a revision of procedure

Gaps in understanding are exposed when some students use a procedure without keeping meaning in mind and others correctly solve the problem because they remain aware of the importance of the meaning of a calculation. This causes conflict, and after reviewing meaning and procedure, a new level of understanding is reached.

Extending situation: What?

As these cases show, due to the fact that the loss of meaning that accompanies procedurization occurs slowly, it is not always possible to differentiate between sections I and II. The major question is how to work towards the climax in section III. In other words, how do teachers teach in order to enable children to overcome the conflict? Looking back on the examples, the following two points, A) and B) must be necessary conditions.

A) Posing tasks which, with poor understanding, will produce different answers.

Tasks should be presented in such a way that there will be a conflict between children who forget or do not care about meaning in acquiring the easy-to-use procedure in section II and children who do keep meaning in mind. In order to do this, tasks must be presented in which children will get stuck or there will be contradictions when easy-to-use procedures are applied in extension situations without due care for meaning. These children may develop their own ideas which should be changed, or they will need to reconsider the meaning.

B) Preparation of meaning that will function as the ground for developmental discussion (a dialectic) and a basis for understanding

For overcoming conflict due to difference in ideas (Hegelian sublation), it is necessary for the children to understand meaning (section I) because this meaning can be used as the basis for the developmental discussion.

In fact, because conflict arises by posing suitable tasks (see part A), or in other words, children encounter results completely different from their own, they are able to ask "What?" or " Why?" This allows them to reflect on their own ideas and take part in developmental discussions as they compare their ideas with those of others. Additionally, the mutual result from this confrontational developmental discussion makes the children produce a response to explain why they arrived at different answers. In the developmental discussion, part B is also necessary. The reason for this is that if the children cannot understand others, or if they cannot accept other's ideas, or if they cannot reproduce other's ideas, their discussion has no common ground as a basis on which to argue and talk at different purposes. If they have a basis for discussion, they can reflect on what others are saying.

When children actually ask each other "Why?", those children who resorted to the easy-to-use procedure (classified as Type 1: 'prioritize procedure without meaning') can do nothing but answer: "Last time 1.5 L was 1L and 5 dL, right? So I did it the same way for 4 m 2 cm," or "You do not make 4 m 10 cm into 4.10 m (in other words, "You do not write it that way") right?" Next, children who correctly applied the meaning to the solution began to talk about the basis (meaning) of the procedure by saying "0.1 m is 1/10 of 1 m, right?" By working out the difference in the meaning of place-value for a dL from the previous time and the relationship between meters and centimeters, the meaning becomes clear. The children who only applied the easy-to-use procedure, and were not conscious of the meaning, now become able to reproduce the correct results. Children who are satisfied with the meaning as discussed are able to revise their own ideas.

3-2. Planning a one-hour class with confirmation of previously learned tasks to reinforce children's knowledge and target tasks

The method indicated for sub-unit construction is also useful for planning a one-hour class. That is, as previously discussed, it demonstrates how to structure a lesson that involves previously learned and target tasks. Here, we will explain Katsuro Tejima's (Joetsu University of Education) introduction to fractions for fourth graders by way of meaning and procedure, and we will show the flow of his lesson structure

(Ref: “Kazu-e-no Kankaku Wo Sodateru Shido,” Elementary School, University of Tsukuba).[5]

Third grade
3/4 Three parts of four equal divisions of a whole.
3/4 m This is composed of three parts of four equal divisions of 1 m.

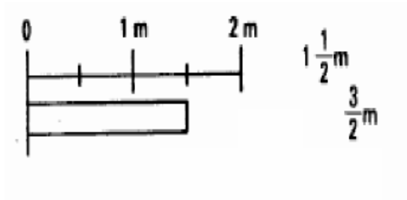
Extending **Extending**

Fourth grade
1 and 1/4 m... This is 1 m and one part of four equal divisions of the other 1 m. 5/4 m This is five parts of four equal divisions of 1 m???

3/4.....Three times 1/4 unit
(Three times one quarter)
5/4.....Five times 1/4 unit
(Five times one quarter)

5/4 m.....Five times 1/4 unit
(Five times one quarter)
→ Collection of unit fraction

First, Tejima revises the meaning of fraction learned in third grade, before improper fractions are introduced in fourth grade (see diagram above). Because “five parts of four divisions of 1 m” makes no sense, it is necessary to teach children about the way of looking at improper fractions as a collection of unit fractions. Also, he tries to utilize the gap between meaning and procedure that occurs in the children’s thinking. In the third grade, even when children study the meaning of “3/4 m is 3 parts of four equal divisions of 1 m”, there are children who learn it as the procedure: “if it is 3/4 m, then take three of the four equal divisions of the whole” because they only learn in the case of equal divisions of the whole. As a result of applying the procedure, 2 m is seen as the whole and the answer is given as 3/4 m.



[5] In Japanese elementary mathematics education, a fraction is first introduced via a situation such as dividing up a pizza or a cake. In this context, it is explained by the part-whole relationship (fraction without denominator). Second, a fraction such as 1/3 m is introduced (fraction with denominator). In this context, the meaning of a fraction is extended from the part-whole relationship to the number line with the idea of a quantity. Thus the improper fraction 4/3 means four lots (four times) one third (one third as a unit fraction). Later, a fraction is recognized as the result of division (for example, the special case of decimal fractions). Finally, a fraction is recognized and interpreted as a ratio. The lesson by Masaki was given based on past curriculum standards (1980). In grade 3, a fraction is introduced as a relation between parts and a whole. Mixed fraction, Improper fraction, Proper fraction, and Unit fraction are taught in 4th grade. The sequence changed a little in 1999 standards.

He used the following structure for a single lesson that incorporates previously learned tasks and target tasks. The aim of the lesson is to bridge the gaps between meaning and procedure that children hold and to clarify misconceptions about the meaning of fractions.

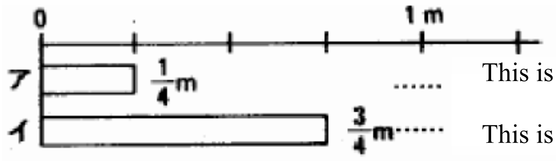
Previously learned task 1: The teacher shows the children a 1 m long piece of tape and divides it
into four parts in front of them. He asks them: “How long is each part?”
C1: 25 cm, C2: 0.25 cm, C3: 4/100 m, C4: 1/4 m

Previously learned task 2: After confirming that the length is expressed as the fraction 1/4 m, the teacher says: “Today, let’s express the length of this tape in fractions.” He then cuts the tape into two pieces: 1/4 m and 3/4 m. As shown below, the teacher then asks: “How can we express lengths

A and B in words? First, let’s think of A as 1/4 m.”

C5 This is one part of four divisions.

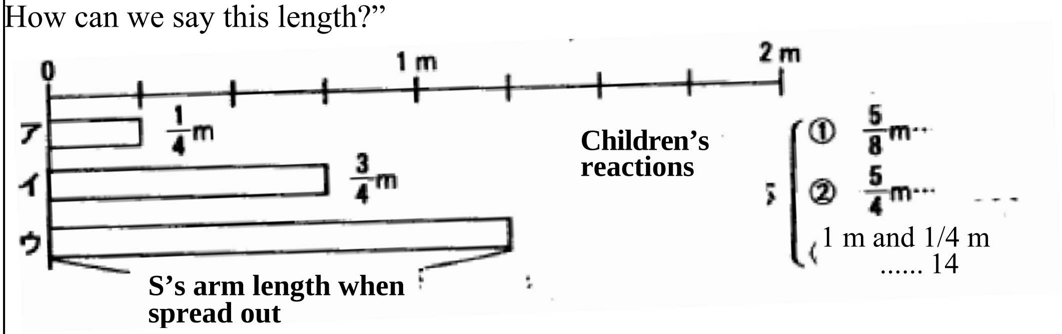
C6 This is one part of four divided parts from 1 m.



This is one part of four evenly divided parts from 1 m.

This is three parts of four evenly divided parts from 1 m.

Target task: Next, the teacher takes out a piece of tape measuring 125 cm. He then says: “T, the length of this tape has a connection to the human body. What do you think it is?” Following this, the teacher develops the discussion by saying: “C, the length of both arms spread out. It is an actual fact.” He then says to the children, as indicated in the diagram below, “When S spreads his arms out, the length is over 1m. How can we say this length?”



The developmental discussion unfolds via a debate about tasks 1 and 2.

C9 I think 5/4 m is strange.

C1 0 It's five parts of the four divisions of 1 m.

C (to C9) That's right./ I disagree.

C11 I disagree. If you take the 1 m away, 1/4 m is left. 1 m equals 4/4 m, so if you put them together, it's 5/4 m.

C1 3 5/4 m is strange because even though 1 m was split into 4 parts, the numerator is bigger than the denominator.

C14 There are one, two, three, four, five lots of 1/4 meters, so it's 5/4 m.

C1 5 If it were 5/8 m, then it would mean it was the fifth part of eight evenly

Summary

If it is 5/8 of 2 m, then that is correct.

If 5/8 m is written with 'm', then it becomes smaller than 1 m, which is strange. It is five times the 1/4 m tape length, so 5/4 m is ok.

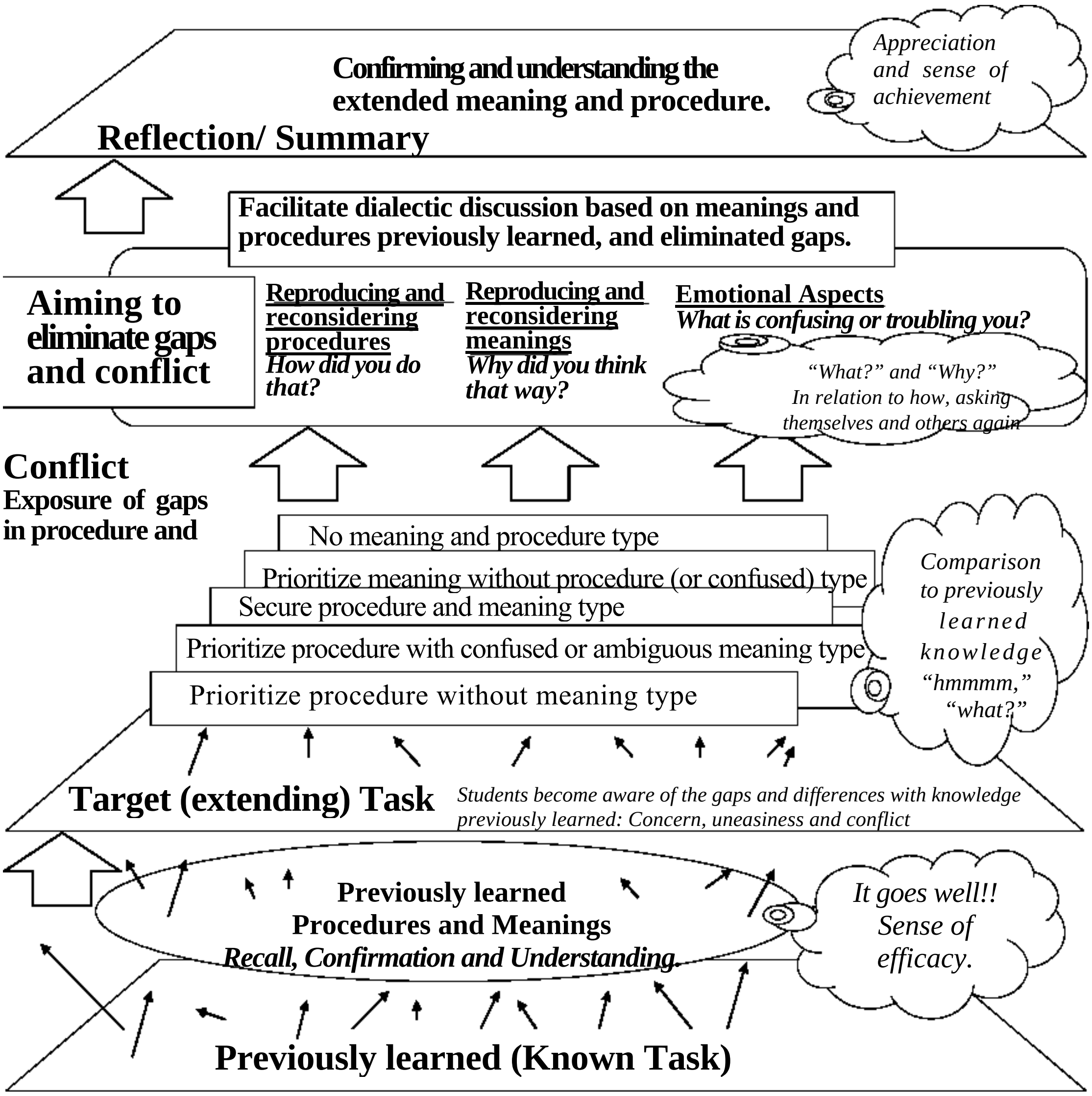
The above is an overview by Tejima. What would have happened if the teacher had begun the class by skipping the review of previously learned material and immediately used the target task? Since the target task is an extension of the previous material, a wide variety of ideas would appear. The developmental discussion would have gone out of control and continued in the same way if children had not shared the grounded meaning of Task 1 (see, Isoda, 1993).

He knows that many children will come up with the answer 5/8 m before he plans the lesson[6]. The goal of this class is to make children aware of a new meaning of multiples of a unit fraction, so that this may serve as a basis for a procedure known as improper fraction representation, which will be covered in the next lesson. To that goal, it is necessary to emphasize to children the idea of aggregating a number of fractions of unit 1/4 m. (Children do not know about a fraction as a unit, or as a number on the number line). At the same time, it is also necessary to revise the misunderstanding of 5/8 m, which comes about from thinking of fractions as equal parts of a whole. In order to revise this idea, he reminds children to consider the length in Task 1 and asks the children if they can confirm that 25 cm = 0.25 m = 1/4 m. In Task 2, he reviews the definition of fractions, confirms it and tests it in the

6 In Japan, the results of lesson studies such as children's ideas in the context of teaching on curriculum sequence have been well shared through teachers' guidebooks and journals. Thus, teachers can expect children's response before the lesson.

target task by placing the 1/4 m and 3/4 m in the tape diagram on a number line in increasing order By creating this contextual flow, it is easy to become aware of “how many 1/4 m parts” there are, such as in the answer 5/4 m. Further, the idea of 5/8, which was obtained without meaning, is “5 parts of 8 equal divisions of 1 m.” This was obtained by applying the previously learned definition of fractional numbers. Children will realize that 5/8 m is smaller than 1 m. Here, counter-examples are effective: “5/8 m is smaller than 3/4 m, so it’s not right.” The developmental discussion was successful, as the meaning and procedure that form the basis of discussion had been confirmed in Task 1 and Task 2 before considering the meaning and procedure in target Task 3.

In conclusion, the lessons of Masaki on parallelism, Suzuki on division and Tejima on fractions can all be summarized as shown in the flow chart below.



In order to run a lesson to include such a flow, the following work (A-D) is necessary for its planning.

A) Investigate which stage of extension this class is at within the curriculum sequence, and what kind of changes are necessary regarding procedure and meaning to achieve the class goals.
B) Consider what types of target tasks are necessary to extend the material.
C) Anticipate what kind of reactions and gaps in meaning and procedure will appear when the children in the class tackle the target task, learned from previous situations.
D) Prepare tasks that review previous material to determine what needs to be covered in terms of meaning and procedure in order to perform the target task. This will also allow the creation of a basis for developmental discussion, which will examine what grounding of meaning is necessary for the elimination of gaps that appear during the target tasks.

If the lesson is developed as a part of a unit or subunit plan for teaching such as Furomoto's lessons on the decimal number, the first part of the lesson flow chart, that is, the previously learned task, can and often is put into the immediately preceding class in consideration of the above, and the lesson usually focuses on the remaining four parts: Target Task, Conflict, Eliminating Gap and Reflection.

3-3. Developmental discussion to eliminate (bridging) gaps

Upon reflection, developmental discussion takes place with the aim of eliminating conflict caused by gaps.

Developmental discussion (a dialectic) that eliminates gaps in diverse ideas

The reactions of children who are no longer aware of the meaning of the procedure	**Conflict**	The reactions of children who remain aware of the meaning of the procedure

Children who are aware of the meaning of the procedure and children who are not aware of it contradict each other. Here, the discussion develops based on the ideas and concerns of children who have an ambiguous understanding of meaning or procedure.

In order to eliminate contradictions and gaps, it is necessary for children to persuade other children to revise their ideas.

Considering what has been discussed so far, it is conceivable that developmental discussion will progress in the expected direction if the following two points are taken into account.

1) Consciously developing “Hmmm” and “Why?”

When children are solving new problems by themselves, they become concerned and uneasy and think “Is it ok to do it like this?” This concern and uneasiness are manifested in children’s feelings when they find gaps in the meanings and procedures of previously learned tasks. However, once the children have successfully answered the task question, they feel better and forget these types of feelings. If children lose the desire to eliminate concern and uneasiness from within themselves, they cannot understand the complex ideas of others. Moreover, they are unable to take note of the viewpoint of others and revise their own ideas by sharing their opinions with their classmates. Children who ‘prioritize procedure with confused or ambiguous meaning’ or ‘prioritize meaning with ambiguous procedure’ often display this type of concern and uneasiness. Therefore, the use of such concerns and uneasiness makes it easy to access the benefits of developmental discussion.

2) Sharing understandings of meanings which will serve as the basis for the developmental discussion

Mutual differences in procedures are exposed as gaps during the developmental discussion. In order to eliminate such gaps, children must talk about the meanings of the basis for each other’s procedures by asking: “Why did you think that way?” In addition, if they do not share or understand each other’s interpretation, they cannot revise their own procedures.

Using the above two points as a premise, the following two points can be shown as measures to set up and summarize developmental discussion.

a) Searching for a mutually recognized meaning to enable children to share a logical explanation as a base.

b) Using other’s ideas even when recognized as inappropriate and deducing contradictions.

It is fundamental for a developmental discussion to be planned with regard to point (a): It is necessary for mathematical explanation as a kind of mathematical proof. However, it is not easy for children to share the meanings. This is because it is difficult to respond when listening to another person’s comments. If children quarrel, a proper debate becomes hard to establish and those involved cannot break away from their own ideas and assertions. Here, the following teaching skills become necessary (see such as Kimiharu Sato, 1995).

- When different ideas are outlined, give children time to reconsider why they think their idea is appropriate, so that they can explain why they think that way.

  Example: Get children to write down their ideas regarding why they think that way.
- Develop the points of confusion and concern as points of discussion in order to organize them within the developmental discussion.

  Example: Ask children to comment on their points of confusion and concern.
- Organize the points of discussion so that arbitrary comments do not cause the developmental discussion to get out of hand.

  Example: “Try to say that again,” “Hold on, I understand what he/she said,” “That’s good.

  Can someone rephrase it?” “Well, the points of discussion are on different levels now. Let me restate the problem.”

Using these teaching techniques, the teacher encourages children to find meaning that everyone is satisfied with and ideas can be presented logically based on this meaning. In such a developmental discussion, point (b) above usually becomes necessary. In the first part of point (b), presuming that ‘the other person is right’ is a necessary condition for considering the other person’s perspective. In other words, what is the premise used to enable children to reach such a result? In order to reach this result, children are required to determine what premises the other children are basing their ideas on. However, it is not an easy task to reproduce another person’s ideas. In actual fact, when performing a task which exceeds the ‘if’ conditions of a procedure that works, it is not uncommon that more than half of the children misconceive the problem and use a procedure without any meaning. Among those children, some answer the way they do because they are unable to understand the reason for that meaning and seek to understand its basis. In that case, even if they listen to another person’s explanation, they cannot agree with the other person’s idea due to the fact that they are unable to understand what the other person is talking about, because they cannot understand the premise on which that person’s idea is based. When this happens, first it is necessary to make the children aware that failing to take the premises into account will cause confusion. A persuasive technique is to suggest that the person temporarily accepts the other’s idea even if it is very different from his/hers, continues to use the idea in another case, and then shows that it will contradict what they already learned before (the latter half of (b)). This is the Socratic dialectical method used since ancient Greek times, and is the origin of the reduction ad absurdum (reduction to absurdity) in terms of mathematics. In simply words, it is the production of a counterexample. If the other person does not understand it as a counterexample, it is not effective. Accordingly, the following section examines two methods that are effective in creating counterexamples.

a. Waiting Counter Example on (b) for (a): What if A’s idea is correct?

Here is an example. Hidenori Tanaka, a teacher at Sapporo Municipal Ishiyama-Minami Elementary School, is teaching fifth graders addition of fractions with different denominators using the example 1/2 + 1/3. Some of the children give the answer 2/5. This answer shows a student in the ‘prioritized procedure without meaning’. These children merely added the numerators and denominators of the fractions together, without understanding the meaning. Further, some children advocated the mistaken meaning by arguing (○●) + (○●●) = (○○●●●) (‘prioritizing procedure with confused or ambiguous meaning’). For children who think this explanation is correct, it shows a lack of understanding of fractions, since it is impossible to add fractions together which are in different units. For this reason, even if the children were able to understand their classmate’s explanation using a diagram, they would not understand why a classmate would say their own diagram explanation was wrong. What disproves their misguided understanding is the rebuttal, “So, have you ever added up denominators before?” According to this procedure, 1/2 + 1/2 = 2/4 = 1/2, and as the children see it, (○) + (○●) = (○○●●). Looking at it this way goes against what has been previously learned. Accordingly, this type of refutation, which is not a straight denial of that person’s idea, uses their answer as an opportunity to critique their way of thinking, and is therefore quite convincing.

b. Facilitating awareness through application of tasks in different situations and examples

The excellent approach of asking “What if A’s idea is correct?” is that it makes use of A’s procedure without meaning. It includes the reasoning based on other’s saying for trying to share the grand of discussion (a). In doing so, it focuses on the contradiction in procedure that the student has used rather than the meaning he or she does not understand. The use of A’s procedure allows him or her to realize his or her own misconception of the procedure. This is the same method seen in Tejima’s class.

However, there are also times when a contradiction needs to be indicated in new tasks in the case, a counter example is not clear for students or not given by students and a teacher does not show it.[7] Here, we present a following example of this method using a third grade fraction class run by Mikiko Iwabuchi, a teacher at Sapporo Municipal Kitasono Elementary School in Sapporo. In this example, a shift from fractions as equal parts of a whole to fractions as quantities on the number line (unit fractions) is planned.

In this planned lesson sequence (see next page), the meaning of fractions as equal parts of a whole is used as a basis for defining fractions as quantities in their own right. This definition creates a shift in meaning from “n parts of m equal divisions of the whole” to “n parts of m equal divisions of a unit quantity.” Up until the second lesson, children have only studied fractions as equal parts of a whole, so there are various discrepancies in the semantic interpretation of the answer as 1/4 m in

[7] If children well educated enabling to change the parameters on the problem by themselves and children have rich custom to explain their idea with the words ‘for example’, posing counter example by children is not rare case in elementary school classroom (See such as Tanaka 2001). Even if there is a child find the counter example against the answer, it is not always understandable for other children.

the third class. The students answers are wide ranging.[8] Debate arises among the children, and as expected, conflict is seen between those who chose answer B and those who chose answer C. In particular, as 1/4 m is read as '1 of 4 parts' m in Japanese, it is easy for the children to arrive at the idea that the number is four times the standard 1 m. As an idea to support C, one child claimed "it should be shorter than the original length" to make use of the meaning studied of fractions as equal parts of a whole. Another is the indication expressed in the comment: "If 1/4 m =1 m, you should say 1 m, otherwise it's strange." However, because the meaning of 1/4 m is undefined and discrepant, the children listening others will not be able to make sense of it. Therefore in the fourth class, the children are asked about the case of 1/2 m by the teacher. If B is correct, 1/2 m = 1 m and 1/4 m = 1m, and so you would have "1/2 m = 1/4 m," which again is strange, and a debate centering on "it should be shorter in the order of 1/2 m, 1/4 m, 1/10 m," would occur from the perspective of what was learned about fractions as equal parts of a whole. In other words, a conclusion that answer C is correct can be reached because the meaning and logic of fractions studied in the second class does not match answer B from the first class.

1st lesson: Halves... dividing equally... introduction of fraction as part-whole relationship using 1/2.

It goes well!

2nd lesson: "Let's make 1/4." Using fraction as parts of a whole. *It goes well!*

The teacher asks children to make a 1/4 size piece of colored paper and tape to send to their sister school, Astor Elementary, for its music festival.

3rd lesson: "Let's make 1/4 m." Introducing fraction as a quantity. *What?*

The teacher wants the children to cut a 1/4 m length of tape to send to their sister school's festival.

They must make sure the measurement is right.

A) The original size of the tape can be any size, so if the whole length is not given, it is not set. (2 children: 'Prioritize procedure with confused or ambiguous meaning.')

B) 4 m is divided evenly, each piece is 1 m. (16 children: 'Prioritize procedure with ambiguous or no meaning.')

C) One piece from 1 m is divided evenly (25 cm). (19 children: 'Secure procedure and meaning type.')

4th lesson: 'Let's make 1/2 m." Introducing fraction as a quantity (continued from the 3rd lesson).

[8] Here, when the meaning matches the definition, it is classified as 'secured meaning; however, as this is at a stage before definition, it does not mean that others are misconceptions.

Based on the above discussion, the second chapter will show the practice of developmental discussion classes that lead to the creation of diverse ideas..

Notes & References in 1996 Japanese version.

From the viewpoint of academic research, the following is an explanation of the research path, its position in mathematics education, as well as the reference materials used in making this book.

In the early 1980s, it can be said that the theoretical framework for the problem solving approach, as it is now known in Japan, had already developed. In actual fact, the contents provided at that time, do not differ much from the research that had been done after constructivism became a significant issue for debate in the mid 1980s. Furthermore, as far as teaching practice is concerned, the level of lessons run by teachers using problem solving techniques in Japan ranks very highly, even from the perspective of constructivists. For example, Jere Confrey (vice-chairperson of International Group for the Psychology of Mathematics Education in 1995, when th book was written), a leader in the field of sublation of radical constructivism and social constructivism, has given a high evaluation of the idea as a constructivist approach in the lessons.

However, in the early 1980s and 1990s, there was a gap. For example, in the early 1980s, the discussion of diverse ideas was in terms of the diversity of correct ideas with open-ended problems. One factor that changed that trend was research about understanding. This chapter has been written to include the way to describe the phases of understanding – conceptual knowledge and procedural knowledge theory – as of the context of research on understanding, as well as to show the theoretical aspects of the problem-solving lesson and teaching practice of teachers from Sapporo.

The following papers act as a framework for this chapter.

Masami Isoda (the author), *"Katto to Nattoku wo Motomeru Mondai Kaiketsu Jugyo no Kozo,"* Riron to Jissen no Kai Chukan Hokokusho, 1991

I have studied much from the following researchers in order to acquire my theory:

Toshio Odaka & Koji Okamoto: *Chugakko Sugaku no Gakushu Kadai.* Toyokan Publishing Co., Ltd., 1982

Tadao Kaneko: *Sansu wo Tsukuridasu Kodomo*. Meijitoshoshuppan Corporation, 1985

Katsuhiko Shimizu: *Sugaku Gakushu ni Okeru Gainenteki Chishiki to Tetsuzukiteki Chishiki no Kanren ni Tsuite no Ichi-kosatsu*. Tsukuba Sugaku Kyoiku Kenkyu, 1989 (co-authored with Yasuhiro Suzuki)

Katsuro Tejima. *Sansuka, Mondai Kaiketsu no Jugyo*. Meijitoshoshuppan Corporation, 1985

J. Hiebert. *Conceptual and Procedural Knowledge: The Case of Mathematics*. LEA, 1986

Toshiakira Fujii. *Rikai to Ninchiteki Conflict ni tsuite no Ichi-kosatsu*. Report of Mathematical Education, 1985

The originality of this book lies in the following areas: applied a descriptive research method of children's understanding in psychology to lesson material and planning; and, applied the viability of knowledge on constructivism to the developing problem situations due to gaps in procedure and meaning that come about from extending and generalization on curriculum sequence.

James Hiebert, who is known as the conceptual and procedural knowledge theory, has appraised these applications.

Below are the references and contents that could not be included in the book although they too are worthy of use in this context.

Author's material:

Sansu Jugyo ni okeru Settoku no Ronri wo Saguru, Kyoka to Kodomo to Kotoba. Tokyo Shoseki Co., Ltd., 1993.

Miwa Tatsuro Sensei Taikan Kinen Ronbun Henshu-iinkai-hen. *Gakushu Katei ni Okeru Hyougen to Imi no Seisei ni Kansuru Ichi-kosatsu, Sugaku Kyouikugaku no Shinpo*. Toyokan Publishing Co., Ltd., 1993.

Sugaku Gakushu ni Okeru Kakucho no Ronri – Keishiki Fueki to Imi no Henyo ni Chakumoku Shite. Furuto Rei Sensei Kinen Ronbunshu Henshu-iinkai. Gakko Sugaku no Kaizen. Toyokan Publishing Co., Ltd., 1995

Mondai Kaiketsu no Shido. Shogakko Sansu Jissen Shido Zenshu 11 Kan, Nobuhiko Noda (Ed). *Mondai Kaiketsu no Noryoku wo Sodateru Shidou*. Nihon Kyouiku Tosho Center, 1995

Kimiharu Sato. Neriai wo Toshite Takameru Shingakuryoku. Kyouiku Kagaku, Sansuu Kyouiku September 1995 issue

In Japanese original version of this book, some words are used with special meanings even if in Japanese. For example, the phrase 'developmental discussion' has been used to describe the aim of restructuring meanings and procedures that children have through dialectical conversations with them. Furthermore, from the standpoints of 'if there is nothing extraordinary, then the idea cannot be truly tried or structured' and 'extending the concept cannot be done without the risk of over-generalization,' we replaced the word 'error (Ayamari in Japanese)' with 'over-generalized idea (Kari but read Ayamari in Japanese)'. This is in line with the meaning of misconception and at the same time is used in the background of an alternative framework on the theory of constructivism.

TEACHERS' MATHEMATICAL VALUES FOR DEVELOPING MATHEMATICAL THINKING TROUGH LESSON STUDY

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1. Mathematical thinking from a sociocultural perspective

Mathematical thinking sounds like an essentially psychological topic. It is just another branch of thinking, and therefore part of the psychological field of knowledge. However, we can never observe mathematical thinking - we can only observe what we assume to be its products, namely mathematical ideas and processes. We can also observe what conditions and contexts might have been responsible for the products of mathematical thinking, which brings us rather closer to the social context.

So what is the problem we are trying to consider here? In one sentence the major problem seems to be: "How can teachers help mathematical thinking to develop in their students?" A subsidiary problem is "How can research on values help with this?" Because of my research work in the field of mathematics education, I prefer to consider mathematical thinking not from a psychological perspective but from a socio-cultural perspective.

2. Three theoretical ideas

In trying to make research progress in solving the problem of helping mathematical thinking develop, I believe we need to consider carefully any theoretical perspectives which might assist us. I will present here three theoretical ideas which I have found helpful in my research and which I believe can shape our understanding of the problem and lead to potential pedagogical solutions. These 'solutions' can then be researched using the Lesson Study method – but more about that later.

2.1 Lancy's developmental theory of cognition

David Lancy (1983) is a cultural psychologist who, in his major cross-cultural study in Papua New Guinea, developed a new stage theory of cognition. It was based on Piaget's theories but he developed them from a socio-cultural perspective. He was doing his research in Papua New Guinea and through investigating cognition with students in PNG, he found that the theoretical developmental sequence of Piaget's stages were similar to, but not identical with, those Piaget found in his European-based research.

He found that Stage 1 was very similar to Piaget's sensory-motor and early concrete operational stages. He argued that this stage is where genetic programming has its major influence, and where socialisation is the key focus of communication. Many activities involving the child are completely similar across cultures.

He then argued that Stage 2, a later concrete operational stage, is where enculturation takes over from socialisation. As he says: “Stage 2 has much to do with culture and environment and less to do with genetics”, and he demonstrated that this is the stage where different cultures will emphasise different knowledge and ideas. Even in relation to mathematics (which is where ethnomathematics develops) this is the case..

The big development in Lancy’s theory from Piaget’s is seen in Stage 3 which concerns the meta-cognitive level. Lancy says: “In addition to developing cognitive and linguistic strategies, individuals acquire ‘theories’ of language and cognition.” Different cultural groups emphasise different ‘theories of knowledge’ and Piaget’s ‘formal operational’ stage is one such theory of knowledge emphasised in Western culture. Confucian Heritage Cultures emphasise other theories of knowledge. These theories of knowledge represent the ideals and values lying behind the actual language or symbols developed by a cultural group.

Thus it is in Stages 2 and 3 that values are inculcated in the individual learners. In a classic work by Kroebner and Kluckholm (1952) they strongly support this idea: “Values provide the only basis for the fully intellible comprehension of culture, because the actual organisation of all cultures is primarily in terms of their values” (p. 340).

Thus for our problem, the idea of mathematical thinking as a form of meta-cognition, affected by the cultural norms and values of the learner’s society, is helpful.

2.2 Billett’s (1998) analysis of the social genesis of knowledge.

But where do these norms, values and knowledge come from, and how can we think about them from a more educational perspective? Stephen Billett’s (1998) sociological work analyses and locates what he calls “the social genesis of knowledge” in 5 inter-relating levels:

Socio-historic knowledge factors affect the values underpinning decisions made by both institutions and teachers. It is knowledge coming from the history and culture of the society, and is value-laden knowledge.

Socio-cultural practice is defined by Billett as historically derived knowledge transformed by cultural needs, together with goals, techniques, and norms to guide practice. At the institutional level these are manifested by curricular decisions influenced by such factors as: (a) current institutional management philosophy with respect to educational and social values (in loco parentis); (b) State or national curricular frameworks and (c) the ethos of the mathematics faculty or teacher’s peer group.

The community of practice in the classroom is identified by Billett as particular socio-cultural practices shaped by a complex of circumstantial social factors (activity systems), and the norms and values which embody them. This community is influenced by (a) the teachers' goals with respect to and portrayal of pedagogical values, (b) students' goals and portrayal of learning values, and personal values.

Microgenetic development is interpreted by Billett as individuals' (teachers' and students') moment-by-moment construction of socially derived knowledge, derived through routine and non-routine problem solving. The nature of teaching as a profession is reflected in the relative autonomy assumed within the walls of the classroom, where teachers' decisions are constantly being made or revised on the basis of a continuous flow of new information. The instantaneous nature of many decisions is likely to be influenced to a greater or lesser extent by the teacher's internalised sets of values.

Ontogenetic development includes individuals' personal life histories, socially determined, which furnish the knowledge with which to interpret stimuli; this development includes participation in multiple overlapping communities.

This analysis points to the different sources of influence on teachers' values. Billett's categorised knowledge is a powerful indicator of how different knowledge at these five levels can impinge on and influence teachers' values in the classroom.

2.3 Bishop's (1988) socio-cultural dimension and its levels

My research context has been in the field of culture, and especially with considering mathematics as a form of cultural knowledge. When we are considering how to develop values in relation to mathematical thinking, I also believe we need to keep in mind the socio-cultural dimension of mathematics education. This dimension influences the values of mathematical thinking at five levels, which are similar but different to Billet's levels.

1. Cultural level – the overarching culture of the people, their language, their mathematics, their core values. In Billet's levels he combined together the cultural and the societal, which I believe in the case of mathematics education is not helpful. Evidence from research at the cultural level shows how different ethnomathematical ideas are not necessarily related to similar societal structures. Ethnomathematics points to cultural influences on mathematical thinking.
2. Societal level – the social institutions of the society, their goals, and their values regarding mathematics. In many societies mathematics education is a contested field with many proponents of different educational 'solutions' vying for publicity and academic advantage. They inevitably affect what is considered to be important mathematical thinking, and who is capable of doing it.
3. Institutional level – the educational institutions' values and the place of mathematics within them. At this level we can see the ways institutional values influence the curriculum, the timetable and even the allocation of space to each subject. These values also affect the development of mathematical thinking in different groups of students.

4. Pedagogical level – the teachers' values and decisions, the classroom culture of mathematical thinking. This is the same level as Billett's 'community of practice', and I have to confess that I prefer Billett's description of this level, as it emphasises the contribution of teacher and students to the classroom knowledge culture.
5. Individual level – individual learners' values and goals regarding mathematics, and mathematical thinking, which can differ markedly, and which do not necessarily follow the teachers' values and goals.

Thus I will draw on these three perspectives in the rest of this talk, and in particular I will assume that my ideas about values regarding mathematical thinking are:

1. Concerned with developing metacognition
2. Located within the socio-cultural dimension
3. Focused on the community of practice in the classroom.

3. Values and mathematical thinking

Now we turn to the values problem stated in Section 1 above. Building on the above analysis, I realised firstly that it was necessary to distinguish between three kinds of values:

- Mathematical values: values which have developed as the knowledge of mathematics has developed within any particular culture.
- General educational values: values associated with the norms of the particular society, and of the particular educational institution.
- Mathematics educational values: values embedded in the curriculum, textbooks, classroom practices, etc. as a result of the other sets of values.

My research approach to values and mathematical thinking has been to focus on mathematical values, and on the actions and choices concerning them (see Bishop, 1988, 1991, 1999). In my work I have used White's (1959) three component analysis of culture:

- Ideological component: composed of beliefs, dependent on symbols, philosophies,
- Sentimental (attitudinal) component: attitudes, feelings concerning people, behaviour,
- Sociological component: the customs, institutions, rules and patterns of interpersonal behaviour.

So how are these components interpretable in terms of mathematical thinking?

3.1 The Ideological component of Mathematical values

In regards to this component of the Mathematical culture, I argued (Bishop, 1988, 1991) that the critical values concern Rationalism and Objectism.
Valuing Rationalism means emphasising argument, reasoning, logical analysis, and explanations, arguably the most relevant value in mathematics education.

Ask yourself as a teacher:
Do you encourage your students to argue in your classes?

Do you have debates?
Do you emphasise mathematical proving?
Do you show the students examples of proofs from history (for example, different proofs of Pythagoras' theorem)?
Valuing Objectism means emphasising objectifying, concretising, symbolising, and applying the ideas of mathematics.

Ask yourself:
Do you encourage your students to invent their own symbols and terminology before showing them the 'official' ones?
Do you use geometric diagrams to illustrate algebraic relationships?
Do you show them different numerals used by different cultural groups in history?
Do you discuss the need for simplicity and conciseness in choosing symbols?

3.2 The Sentimental (Attitudinal) component of Mathematical values

In regards to this component, the important values are Control and Progress.

Valuing Control means emphasising the power of mathematical and scientific knowledge through mastery of rules, facts, procedures and established criteria.

Ask yourself:
Do you emphasise not just 'right' answers, but also the checking of answers, and the reasons for other answers not being 'right'?
Do you encourage the analysis and understanding of why routine calculations and algorithms 'work'?
Do you always show examples of how the mathematical ideas you are teaching are used in society

Valuing Progress means emphasising the ways that mathematical and scientific ideas grow and develop, through alternative theories, development of new methods and the questioning of existing ideas.

Ask yourself:
Do you emphasise alternative, and non-routine, solution strategies together with their reasons?
Do you encourage students to extend and generalise ideas from particular examples?
Do you stimulate them with stories of mathematical developments in history?

3.3 The Sociological component of Mathematical values

In regards to this component, the important values are Openness and Mystery.

Valuing Openness means emphasising the democratisation of knowledge, through demonstrations, proofs and individual explanations.
Ask yourself:

Do you encourage your students to defend and justify their answers publicly to the class?
Do you encourage the creation of posters so that the students can display their ideas?
Do you help them create student math newsletters, or web-pages, where they can present their ideas?

Valuing Mystery means emphasising the wonder, fascination, and mystique of mathematical ideas.
Ask yourself:
Do you tell them any stories about mathematical puzzles in the past, about for example the 'search' for negative numbers, or for zero?
Do you stimulate their mathematical imagination with pictures, artworks, images of infinity etc.?

These then are what I believe to be the crucial values underpinning the development of mathematical thinking in the classroom. I think we will make good progress in solving our problems if more research is devoted to investigating ways of developing these values in our teachers, so that they can develop them in their students.

4. Values, Mathematical Thinking and Lesson Study

Researching values development is no easy matter, but Lesson Study is an excellent method for studying the development of values in the classroom. In our Values and Mathematics Project (VAMP) we already used a version of lesson study, but without trying to affect the teachers' plans for their lessons.

1. The teachers told us before the lessons what values they thought they were going to develop.
2. We observed and recorded the lessons
3. We interviewed the teachers after the lessons to have them explain what they thought they had achieved.

More details of this research can be found at:
http://www.education.monash.edu.au/research/groups/smte/projects/vamp/vamppublications.html

For a full lesson study of mathematical thinking values, it would be necessary to plan together with the teachers what values they would try to develop.
The teaching ideas earlier would be very appropriate for this. It would be important for the experiment to go over a group of lessons, as values could hardly be developed in one lesson.

5. Conclusions for research

1. With any design and development research it is essential to have good theories to support and structure the work
2. Mathematical thinking has been studied in many ways, but in relation to values it is important to consider it as an aspect of meta-cognition.

3. The context for the research should be the classroom, as it is there that the community of practice significantly influences the meta-cognitive aspects of mathematical thinking.
4. Equally important to consider in this research is the socio-cultural context, as any educational values are embedded in the culture of the society.
5. Lesson study is an excellent research approach for studying any experimental educational development.
6. It is particularly appropriate for studying values development.
7. However there need to be a series of lessons studied as values do not develop in the space of one lesson.
8. Finally the teachers need special support in this research, as values teaching involve the teacher's pedagogical identity, which must be respected (Chin, Leu & Lin, 2001).

References

Billett, S. (1998). Transfer and social practice. *Australian and New Zealand Journal of Vocational Education Research*, 6(1), 1-25.

Bishop, A.J. (1988) *Mathematical enculturation: a cultural perspective on mathematics education*. Dordrecht: Kluwer

Bishop, A.J. (1991) Mathematuical values in the teaching process. In A.J.Bishop, S.Mellin-Olsen & J. van Dormolen, *Mathemaical knowledge: its growth through teaching* (pp 195-214) Dordrecht, Holland: Kluwer

Bishop, A. J. (1999). Mathematics teaching and values education — An intersection in need of research. *Zentralblatt für Didaktik der Mathematik, 1/99*, 1-4.

Chin, C., Leu, Y.-C., & Lin, F.-L. (2001). Pedagogical values, mathematics teaching, and teacher education: Case studies of two experienced teachers. In F.-L. Lin & T.

J. Cooney (Eds.), Making sense of mathematics teacher education (pp. 247-269). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Kroebner, A.L. and Kluckholm, D. (1952) *Culture; a critical review of concepts and definitions*. New York: Vintage books

Lancy, D.F. (1983) *Cross-cultural studies in cognition and mathematics*. New York: Academic Press.White, L.A. (1959) *The evolution of culture*. New York: McGraw Hill

TEACHERS' MATHEMATICAL THINKING

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Introduction

In my presentation at the Tokyo 2006 APEC symposium I demonstrated that mathematical thinking is important in three ways.

- Mathematical thinking is an important goal of schooling.
- Mathematical thinking is important as a way of learning mathematics.
- Mathematical thinking is important for teaching mathematics.

I spent most of that presentation discussing the first two dots points, and only discussed the third point with one example. In this presentation, I will discuss the third point in more depth. I ended my presentation at the last symposium with these comments:

"For those us who enjoy mathematical thinking, I believe it is productive to see teaching mathematics as another instance of solving problems with mathematics. This places the emphasis not on the static knowledge used in the lesson as above but on a process account of teaching. In order to use mathematics to solve a problem in any area of application, whether it is about money or physics or sport or engineering, mathematics must be used in combination with understanding from the area of application. In the case of teaching mathematics, the solver has to bring together expertise in both mathematics and in general pedagogy, and combine these two domains of knowledge together to solve the problem, whether it be to analyse subject matter, to create a plan for a good lesson, or on a minute-by-minute basis to respond to students in a mathematically productive way. If teachers are to encourage mathematical thinking in students, then they need to engage in mathematical thinking throughout the lesson themselves."

The first announcement for the December 2006 Tokyo APEC conference states that a teacher requires mathematical thinking for analysing subject matter (p. 4), planning lessons for a specified aim (p. 4) and anticipating students' responses (p. 5).These are indeed key places where mathematical thinking is required. However, in this section, I concentrate on the mathematical thinking that is needed on a minute by minute basis in the process of conducting a good mathematics lesson. Mathematical thinking is not just in planning lessons and curricula; it makes a difference to every minute of the lesson. In this analysis, I aim to illustrate how strong and quick mathematical thinking provides the teacher with many possible courses of action. The course of the lesson, though, is then determined by how the teacher weighs up the possibilities which he or she sees. The mathematical possibilities are considered along with knowledge of students' mathematical understandings and needs and with pragmatic factors (eg those associated with keeping the lesson on track), and a choice is made. These decisions determine the course of a lesson.

We now examine the mathematics used by two teachers when their classes tackle the 'spinners game'. After this, I also report on experiences when the problem was adapted and used in a primary teacher education class.

Irene's lesson on the Spinners Game

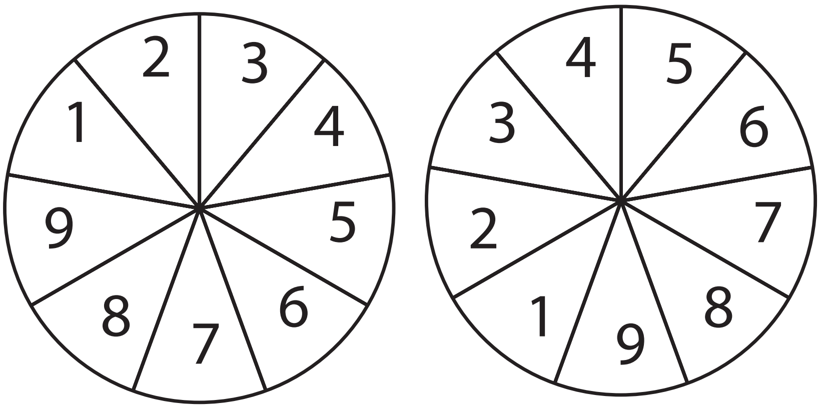

Figure 1. Equipment for the spinners game

The spinners game was first discussed in Chick and Baker (2005) and is based on their classroom observations and interviews with the teachers. This account of two classroom uses is reproduced with adaption from Chick (2007) with permission, and additional points relevant to this presentation have been inserted. Irene, an experienced teacher, and Greg, who was in only his second year of teaching, were Grade 5 teachers in the same school. They had chosen to use a spinner game suggested in a teacher resource book (Feely, 2003). The spinner game used two spinners divided into nine equal sectors, labelled with the numbers 1-9 (see Figure 1). The worksheet instructed students to spin both spinners, and add the resulting two numbers together. If the sum was odd, player 1 won a point, whereas player 2 won a point if the sum was even. The first player to 10 points was deemed the winner. Students were further instructed to play the game a few times to "see what happens", and then decide if the game is fair, who has a better chance of winning, and why (Feely, 2003, p. 173). The teacher instructions (Feely, 2003, p. 116) included a brief suggestion about focusing on how many combinations of numbers add to make even and odd numbers but did not provide any additional direction.

This game can offer worthwhile learning opportunities associated with sample space, fairness, long-term probability, likelihood, and reasoning about sums of odd and even numbers. The significant issue here, especially in the absence of explicit guidance from the resource book is *how* these learning opportunities can be brought out. Although it is not written in the teachers' resource book, the spinners game has an interesting twist. Analysis of the sample space shows that the chances of Player 2 (even) winning a point is 41/81 compared to 40/81 for Player 1 (odd). Player 2 is therefore theoretically more likely to win, however this miniscule difference in likelihood implies that the game's theoretical unfairness will not be evident when playing "first to ten points". We cannot tell whether the authors of the resource book chose this narrow difference deliberately or accidentally. Our interest here is in the teachers' mathematical thinking as they implemented the activity in the classroom.

Irene started the spinners' game late in a lesson. Most students had played the game for a few minutes before Irene began a short class discussion. She asked the class if they thought it was a fair game. Discussion ensued, as students posed various ideas without any of them being completely resolved. For instance, someone noted that fairness requires that players play by the rules of the game. Most of the arguments about fairness were associated with the number of odds and evens, both in terms of the individual numbers on the spinners (there are 5 odds and only 4 evens on each spinner) and in terms of the sums. One student neatly articulated an erroneous parity argument, that since "odd + odd = even and even + even = even but odd + even = odd, therefore Player 2 has two out of three chances to win". Irene said she was not convinced about the "two out of three", but she agreed the game was unfair.

The student's presentation of this argument, which Irene suspects is not valid, requires her to make a decision as to whether it should be pursued, or passed over quickly in favour of something else. She might, for example, have presented (or sought from a student) another erroneous argument along the same lines but which takes into account the fact that odd + even and even + odd occur in different ways: "odd + odd = even and even + even = even but odd + even = odd *and even + odd = odd*, therefore *both* players have two out of *four* chances to win". Presenting this argument would have emphasised that the different orders are important, but the new argument has the same failing as the first argument. It does not take into account that there are different numbers of odd and even numbers. Instead Irene might have decided to highlight just this failing of the student's argument, showing, for example that odd + odd is more likely than even + even. The several possibilities for responding to the argument as well as the possibility of simply passing over it quickly, as she chose to do, must be identified and evaluated in just a few seconds as the classroom discussion proceeds.

Good decisions would seem to be enhanced when teachers see the mathematical possibilities quickly and evaluate them from a mathematical point of view (what important mathematical principles/processes/strategies/attitudes would the students learn from this). However, decision making also needs to be informed by knowledge of how the students will respond, and by attention to practical aspects of the lesson, including the time available. For Irene, the necessity to finish the spinners game in the few remaining minutes of the lesson might have been the over-riding consideration.

Irene then allowed one of the students to present his argument. At the start of the whole class discussion this student had indicated that he had not played the game at all but had "mathsed it" instead, and at that time Irene made a deliberate decision to delay the details of his contribution until the other students had had their say. He proceeded to explain that he had counted up all the possibilities, to get 38 even totals and 35 odd totals. Although this was actually incorrect, Irene seemed to believe that he was right and continued by pointing out that this meant that “it’s [the game is] not *terribly* weighted but it *is slightly* weighted to the evens”. Irene then asked the class if their results bore this out, and highlighted that although the game was biased toward Player 2 this did not mean that Player 2 would always win.

As suggested earlier, the spinner game provides the opportunity to examine sample space, likelihood, and fairness. Given the impact of time constraints on Irene’s lesson, sample space was not covered well, although she believed that the student who had “mathsed it” had considered all the possibilities. This highlights a contrast between her knowledge of his capabilities and of the details of the content with which he was engaged. On the other hand, her content knowledge was sufficient for her to recognise the significance of the small difference between the number of odd and even outcomes and its impact on fairness. Irene led a good discussion of the meaning of fairness and the magnitude of the bias, and its consequence for the ‘first to ten’ aspect. Given the short time available to end the lesson, it may have been a wise decision to ignore the errors in the student’s sample space and go on to what Irene probably saw as the main point: that the difference in likelihood is very small, and that even if there is a bias students would not have been able to reliably detect it in the ‘first to ten’ game.

In considering Irene’s lesson, we see that its path is determined by many small decision points: who to call on next, whether to check the student’s list of outcomes or simply believe him because he is a good student, whether to pursue the errors in the parity argument etc. These decisions are influenced by factors relating to the mathematics (as perceived during the flow of the lesson by the teacher), factors relating to the students’ current knowledge and factors relating to the pragmatic conduct of the lesson (e.g. how much time is left). This is illustrated in Figure 2.

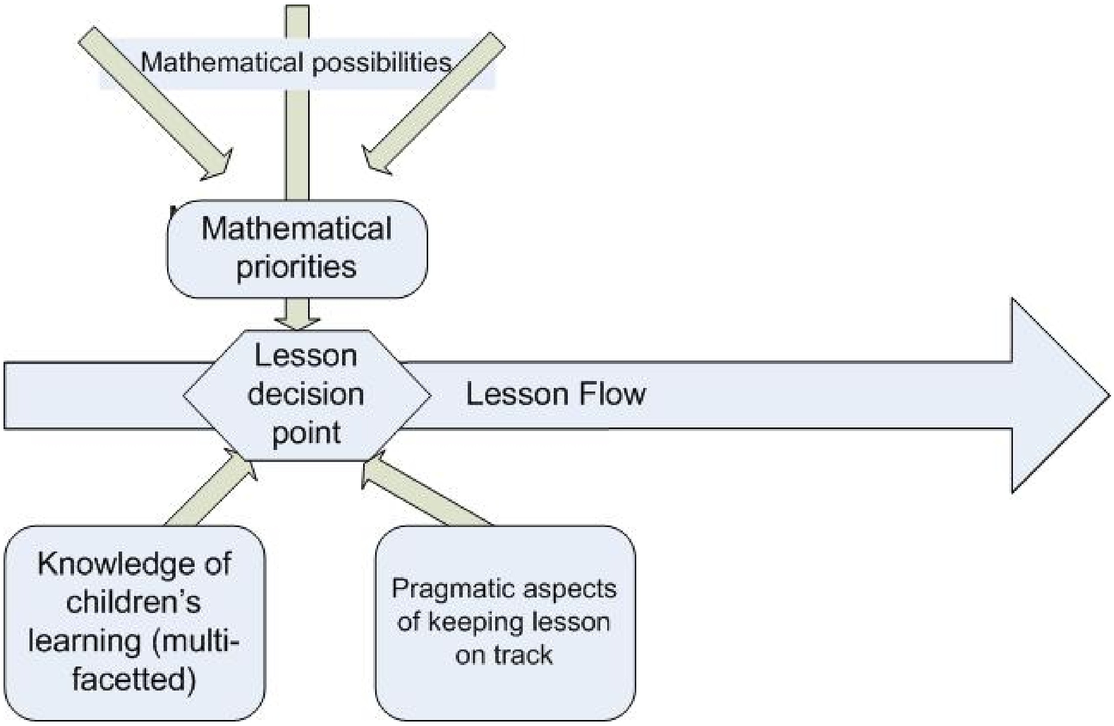

Figure 2. *Decision points in lessons are influenced by many factors.*

Greg's lessons on the Spinners Game

In Greg's class, students played the spinners game at the end of a lesson, and students put forward various ideas about whether the game was fair. During this time, Greg decided that the next lesson should be spent on finding the sample space. Greg then devoted nearly half of his second lesson to an exploration of the sample space. As reported in Chick and Baker (2005) he tightly guided the students in recording all the outcomes and could not deal with alternative approaches. He asked the students to calculate the probabilities of particular outcomes, which was helpful in highlighting the value of enumerating the sample space, but detracted from the problem of ascertaining whether even or odd outcomes were more likely. Students eventually obtained the "40 odds and 41 evens" conclusion, at which point Greg stated that because the "evens" outcome was more likely the game was unfair. There was, however, no discussion of the narrowness of the margin, or the difficulty of confirming this result empirically through the 'first to ten' aspect of the game.

In summary, Greg was much more thorough than Irene in his consideration of sample space, but also very directive. Neither teacher seemed aware of all that the game afforded in advance of using it, as evidenced by the way it was used, although Greg recognised the scope for examining sample space part way through the first lesson. Both teachers were, however, able to bring out some of the concepts in their use of the game, with Irene having a good discussion of the meaning of fairness and the magnitude of the bias, and Greg illustrating sample space and the probability of certain outcomes. An important observation needs to be made here. The teacher guide that was the source of the activity gave too little guidance about what the spinner game afforded and how to bring it out. Even if such guidance had been provided, there is also still the miniscule bias problem inherent in the game's structure that affects what the activity can afford. It is very difficult to convincingly make some of the points about sample space, likelihood, and fairness with the example as it stands. It can be done, but the activity probably needs to be supplemented with other examples that make some of the concepts more obvious (see, e.g., Baker & Chick, 2007). This highlights the crucial question of how can teachers be helped to recognise what an example affords and then adapt it, if necessary, so that it *better* illustrates the concepts that it is intended to convey.

Interestingly, in both classes the students did not—indeed could not in any reasonable time frame —play the game long enough for the slight unfairness to be genuinely evident in practice, yet most students claimed that the game was biased towards even. This may have occurred because the incorrect parity argument made them more aware of the even outcomes than the odd ones.

The observers were surprised by the tight way in which Greg controlled the method by which the outcomes were enumerated. He wanted to see the 81 outcomes, along the lines of the enumeration on the left hand side of Figure 3, although in an array setting out. Greg seemed constrained by his mathematical knowledge, having only one way to think of the sample space—via exhaustive enumeration. When a student offered an erroneous suggestion which could have been readily adapted to a more elegant and insightful method, he did not encourage or discuss it. In fact, there are many bridges between the totally routine method of writing out 81 outcomes and counting how many totals are even or odd, and insightful ways which give the answer quickly. At the top right hand side of Figure 3, for example, is one of the bridges. As they begin work on the exhaustive enumeration on the left hand side, students might be encouraged to note the patterns – alternating evens and odds for a fixed first choice, the EOEOEOEOE pattern when the first choice is odd and the OEOEOEOEO pattern when the first choice is even. These patterns are easily explained by students, and they can be readily utilised to find the how many even and odd sums there are, either by addition or by multiplication as outlined in the figure. The tree diagram approach at the bottom of Figure 3 would be too sophisticated for Greg's young students, since it relies on more strongly combinatorial thinking, but a version of it might be reached after experience with the patterns above.

Enumeration of 81 separate outcomes			
First	**Second**	**Sum**	**Sum**
1	1	2	E
1	2	3	O
1	3	4	E
1	4	5	O
1	5	6	E
1	6	7	O
1	7	8	E
1	8	9	O
1	9	10	E
2	1	3	O
2	2	4	E
2	3	5	O
2	4	6	E
2	5	7	O
2	6	8	E
2	7	9	O
2	8	10	E
2	9	11	O
3	1	4	E
3	2	5	O
3	3	6	E
3	4	7	O
3	5	8	E
etc	etc	…	…

Enumeration mixing number and parity considerations					
First	**Second**	**Sum**	**First**	**Second**	**Sum**
Odd	1	E	Even	1	O
(eg 1)	2	O	(eg 2)	2	E
	3	E		3	O
	4	O		4	E
	5	E		5	O
	6	O		6	E
	7	E		7	O
	8	O		8	E
	9	E		9	O
5 evens and 4 odds			4 evens and 5 odds		
The 5 odd numbers contribute 25 evens and 20 odds, and the 4 even numbers contribute 16 evens and 20 odds, making 41 evens and 40 odds altogether.					

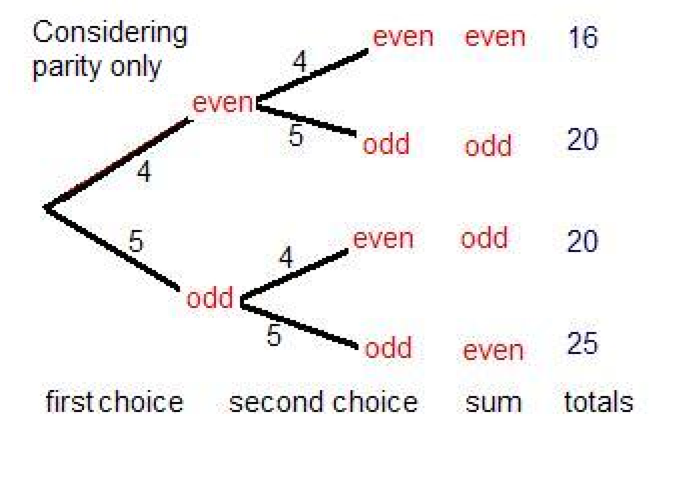

Figure 3. *Three different ways of counting numbers of odd and even spinner totals*

In considering why Greg made his decision to focus his lesson on finding the 81 element sample space in one particular way, it is again likely that his decision is influenced by judgements about mathematical factors, factors related to the students and their current knowledge and pragmatic factors related to the lesson. Greg decided in the first lesson that he would allocate the second lesson to finding the sample space, so it was a priority for him, and he taught it thoroughly. Whereas Irene's treatment of sample space appeared rushed in response to a shortage of time, Greg decided that this was sufficiently important for a second lesson. His priority given to the idea of sample space is also evident in the observation that he did not focus only on the spinner game, but used the sample space to find the probability of events unrelated to the initial spinner game.

As is illustrated in Figure 2, mathematical priorities can only be chosen from the mathematical possibilities that are perceived by the teacher. Consequently, it may be that Greg's focus on one way of finding the sample space was because he was not aware of other ways, or was uncertain of their validity. On the other hand, it may have been a more active prioritising. He may have seen value in teaching students about systematic listing, and wanted students to go through that process very thoroughly, getting a real 'feel' for how to go through the cases one by one. From yet another point of view, Greg may have judged that the full, very routine, case-by-case enumeration was at an appropriate level for his target group in that class, and so he may have selected the method as optimal for the whole class, even if not for each individuals.

This is all speculation, even though Greg was interviewed about his lesson (which contained many other features). It is simply not possible for teachers to thoroughly explain each of the myriad decisions that are made in the course of any one lesson. The point of this discussion, though, is that at any stage in the lesson, Greg was aware of certain mathematical possibilities. These may have resulted from deep or superficial insight into the spinners game; they may be numerous or sparse; they may be mainly procedural or extend to strategic thinking etc. To make a decision on how to respond to a student's question or a mathematical problem arising in the conduct of the class, Greg has to set priorities and act on them. In this way, we see that a teacher's mathematical knowledge (conceptual, procedural, strategic etc) sets the choices and so is very important, but good decision making also depends on teachers being able to make good choices amongst them, in the light of progressing the main aims of the lesson.

Helen's lesson on the spinners game

Even when lessons are videotaped and teachers are interviewed after the lesson, much of the mathematical thinking upon which teachers make decisions about the paths of lessons remains hidden. For this reason, the next example is about a discussion on an episode in a teacher education class, which we discussed together on several occasions.

Helen teaches pre-service primary teacher education students and is a highly accomplished mathematical thinker. She had observed the lessons of Irene and Gary, and decided to use the spinners game in class. She wanted student teachers to analyse what mathematical learning it could generate and how. To simplify and also to extend the game, Helen changed the numbers on the two spinners (see Baker and Chick (2007) for examples).

On one occasion, Helen's class used two spinners labelled with 0, 1, 2 and 3. This small change, selected by Helen to simplify the game, caused a new complication. Many of her students began the enumeration, but halted when they needed to decide whether the sum of 0, obtained by throwing 0 on each spinner, was an even number or an odd number, both or neither. This turned what was intended to be a short mathematical episode using a simplified spinners game used, to an unpredicted query about odd and even numbers.

At this point, Helen faces a decision. Once again it will be informed by her knowledge of mathematics and her mathematical thinking during the lesson, and by weighing the priorities for the lesson. This will be discussed below. However, it is worth observing first that Helen had not predicted the evenness of 0 would be such an obstacle to the progress of this lesson. In future use of the spinners game, having this additional knowledge of students (urther pedagogical content knowledge), she may avoid using the number 0 on the spinners so that the lesson proceeds without this obstacle, or she may deliberately choose it to uncover these misconceptions.

Addressing the apparently simple question of whether 0 is even or odd or neither or both, draws again on mathematical knowledge and pedagogical content knowledge (Shulman, 1986, 1987) working in tandem. The student teachers were very familiar with the fact that 2, 4, 6, 8, 10, 12, etc are even numbers. Why would they query whether 0 is even, and what would convince them that it is? Possibly the reason for the difficulty is that students draw on intuitive meanings for ‘even’, rather than a mathematical definition. For example, they may associate an even number with the possibility of pairing up. If there is an even number of children in our class, we can go for a walk arranged in pairs. If there is an odd number of children, there will be one left over, as illustrated in Figure 4.

Figure 4. *An even number of children can walk in pairs, but not an odd number.*

This informal interpretation of ‘even number’ is difficult to apply to decide whether 0 is even or odd, because whilst there is certainly not ‘one left over’, there are no pairs either. Kaplan (1999) discusses difficulties like this. Alternatively, students who draw just on the list of examples to decide if a number is even or odd (2, 4, 6, … etc) have no way of knowing whether 0 should really be on the list or not, when there is no principle to guide them. They know 0 is special – is this another way in which it is special?

Helen was keen to draw her students’ attention to the mathematical definition of an even number, but she reported that she immediately saw two possibilities. She could say that an even number is defined to be an integer which is exactly divisible by 2 or that an even number is defined to be an integer that is equal to 2 times an integer. This might seem a small difference, but Helen chose the second version because of her previous experience with the awkwardness in teaching associated with discussions of dividing *by* zero. Even though the test for evenness does not involve dividing *by* zero (but dividing *by* 2), Helen avoided the division explanation because she felt students may confuse the situations. In other words, she presented students with finding whether there is an integer satisfying the first rather than the (equivalent) second equation below:

$$0 = 2 \times ?$$

$$0 \div 2 = ?$$

[In neither case could she avoid the likely obstacle of students' uncertainly about whether 0 is an integer.] Here we see that Helen's strong mathematical knowledge and her ability to see the mathematical possibilities quickly presented her with possibilities. Her pedagogical content knowledge (in this case of likely students' difficulties) guided her choice.

Was it best to pause to discuss why 0 is even? Helen could have just asserted that 0 is even and moved the lesson back on the track of investigating the fairness of the spinners game. When reflecting on this question, Helen asserted that the diversion was useful because it enabled her to clarify some fundamental misunderstandings about zero and to show how mathematical concepts are determined by definitions. Here, we see that Helen justifies her choice in terms of her understanding of important principles of doing mathematics – in this case the role of definitions in mathematics. More fundamentally, it seems to reveal a predisposition on Helen's part to avoid having students see mathematics as arbitrary and without reason.

After her observations of the lessons of Irene and Greg, Helen and her colleague published a suggested teaching sequence for primary classes using the spinners game (Baker and Chick, 2007). The spinners she suggests have no zeros. Her suggested sequence begins with a pair of spinners each with just 3 digits, arranged so that there is a strong enough bias to be evident in empirical trials. Students begin by finding this empirical experience of the bias, tallying class results. Students then draw up the sample space and compare theoretical probabilities to empirical class results. They discuss variations between theory and experiment. The pair of spinners chosen are biased towards odd totals (they do not have the same numbers on each spinner – see mathematical note below. Helen has selected these spinners so that the false parity arguments give an obviously wrong answer. This is a very substantial example of mathematical thinking being used in lesson planning, again in concert with pedagogical content knowledge – in this case knowledge of students' false arguments. Helen's suggested lesson sequence then moves back to the original spinners problem. this gives experience in finding a large sample space systematically and subtleties of comparing theoretical and empirical results when the bias is small. Finally students create their own spinners and discuss what they designed the spinners for, how unfair the game is, and what is likely to happen if they play the game many times.

Conclusion

At the beginning of this paper, I drew an analogy between teaching a mathematics lessons and solving a real world problem with mathematics. I noted that in order to use mathematics to solve a problem in an area of application, mathematics must be used in combination with knowledge from the area of application. In the case of teaching mathematics, the area of application is the classroom and so the teacher as 'mathematical problem solver' has to draw on general pedagogy as well as mathematical pedagogical content knowledge to contribute to the solution. As will many problems in areas of application of mathematics, these teaching problems need to be solved in an environment that is rich in constraints: short lesson times, inadequate resources at hand, etc. In the teachers' role of analysing subject matter, designing curricula or in creating a plan for a good lesson, solving the problem can occur with adequate time for reflection, testing ideas and reconsidering choices. However, in the course of a lesson, this mathematico-pedagogical thinking happens on a minute-by-minute basis, with the aim of responding to students in a mathematically productive way. If teachers are to encourage mathematical thinking in students, then they need to engage in mathematical thinking throughout the lesson themselves, but this mathematical thinking is under severe time pressure.

In the conduct of a lesson, teachers see various mathematical possibilities. Some teachers will see more than others in any given situation and some of the possibilities that teachers see may not be correct. The process of choosing amongst these possibilities, which again occurs on a minute by minute basis, will be guided by the deep knowledge of the students (the actual current mathematical knowledge of these students as well as thinking typical of students like these), operating under the constraints of teaching a lesson in a fixed time to achieve an identified goal. Teachers who are stronger mathematical thinkers will see more possibilities, and in the moment when a decision needs to be made, their choices will be better informed by teaching underlying mathematical processes and strategies.

A mathematical note

Solving the problem of bias in the spinners game is a nice example in algebraic factorisation, with surprising results.

If there are n even numbers and m odd numbers on the spinners, then there are $n^2 + m^2$ ways of getting an even total, and $2mn$ ways of getting an odd total (see Figure 5). Since $n^2 + m^2 - 2mn = (n-m)^2 \geq 0$ we can conclude that
(i) if $n = m$ then the spinner game is fair
(ii) otherwise, there is always slightly more chance of getting an even number.

Moreover, if the numbers on the spinners are consecutive whole numbers, then n and m will either be equal or differ by 1 (ie n –m = 0 or |n-m| = 1). This means that the number of even sums will always be equal to, or one more than the number of odd sums. In this way, we see that the very close comparative probabilities of the original spinners game (41/81 and 40/81) are typical of having consecutive numbers on the spinners.

To generalise further, if there are n_1 evens and m_1 odds on the first spinner and n_2 and m_2 on the second spinner (respectively) then there are n_1n_2 + m_1m_2 even sums and n_1m_2 + n_2m_1 odd sums. Are evens or odds more likely to be thrown? Calculate the difference in number of outcomes:

$$n_1m_1 + n_2m_2 - (n_1m_2 + n_2m_1) = (n_1 - m_1)(n_2 - m_2)$$

This means that if evens are more prevalent on both spinners OR odds are more prevalent on both spinners (ie the two factors in the final product have the same sign), then the game is biased in favour of the even sums. Alternatively, it evens are more prevalent on one spinner and odds more prevalent on the other spinner (ie the two factors in the product have opposite signs), then the game is biased in favour of odds.

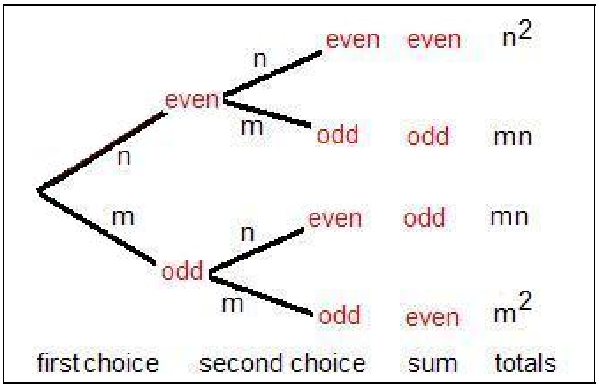

Figure 5. *Odd and even spinner totals from spinners with n even and m odd numbers*

References

APEC –Tsukuba (Organising Committee) (2006) *First announcement. International Conference on Innovative Teaching of Mathematics through Lesson Study.* CRICED, University of Tsukuba.

Baker, M., & Chick, H. L. (2006). Pedagogical content knowledge for teaching primary mathematics: A case study of two teachers. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 60-67). Sydney: MERGA.

Baker, M., & Chick, H. L. (2007). Making the most of chance. *Australian Primary Mathematics Classroom*, 12(1), 8-13.

Chick, H. L., & Baker, M. (2005). Teaching elementary probability: Not leaving it to chance. In P. C. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia*, pp. 233-240). Sydney: MERGA..

Chick, H.L. (2007) Teaching and Learning by example. In J. Watson & K. Beswick (Eds), *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia*, Hobart, Tasmania: MERGA.

Feely, J. (2003). *Nelson maths for Victoria: Teacher's resource Year 5*. Melbourne: Thomson Nelson.

Kaplan, R. (1999) *The Nothing That Is: A Natural History of Zero.* London: Penguin

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Shulman, L.S. (1986) Those who understand: Knowledge growth in teaching, *Educational Researcher* 15 (2), 4-14.

Shulman, L.S. (1987) Knowledge and teaching: Foundations of the new reform, *Harvard Educational Review* 57(1), 1-22.

Stacey, K. (2007) What is mathematical thinking and why is it important? *APEC Symposium. Innovative teaching mathematics through lesson study II*. 3-4 December 2006.

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MATHEMATICAL THINKING IN JAPANESE CLASSROOMS

Abraham Arcavi

Introduction: personal background and intention

As a young student, I have experienced mathematics classrooms at the elementary and secondary school level and at the university level in Argentina. After that, I have experienced mathematics classrooms as a graduate student in Israel. In both countries, I taught mathematics at junior high school and high schools. In Israel and the USA, I was engaged in curriculum development, teacher education and research on teaching and learning. It is against that variegated, yet all-Western, background that I was exposed to the fascinating microcosms of Japanese mathematics classrooms at the elementary school level. Before I went to CRICED, at Tsukuba University to work with Professor Masami Isoda for four months, I had read research describing lesson study in Japan (e.g. Fernandez and Yoshida, 2004, and other sources) which focused on comparing and contrasting Japanese (and other Asian classrooms) with American classrooms (e.g. Stevenson and Stigler, 1992, Stigler and Hiebert, 1999). I had also watched and analyzed an algebra and a geometry lesson (published by TIMSS in 1999), and on their basis, an in-service workshop for teachers and teachers of teachers was designed and implemented (Arcavi & Schoenfeld, 2006) drawing on ideas from the Teacher Model developed by Schoenfeld (1998). However, I think I was able to fully appreciate the teaching in mathematics classrooms in Japan only through the non-mediated experience (except for the simultaneous translation) of “being there”, watching how lessons evolve, following children’s work and discussions, talking to teachers and researchers and sensing the common pedagogical and mathematical characteristics of all the lessons I saw.

If we take the statement that “there is nothing more practical than a good theory” (Lewin, 1952, p. 169) and attempt to formulate its symmetrical version, we may propose that “there is nothing more theoretical than a good practice”. This may make little sense as stated, however, it may suggest that a exemplary practice can be a powerful source for theorizing, which in turn may help understand the practice, especially a teaching practice. Our field has many learning theories, however there are not so many instructional theories. It is with the intention of contributing to instructional theories, that I would like to briefly share what I have learned from mathematics classrooms in Japan.

Mathematical Thinking – A Japanese view

“Mathematical thinking is the “scholastic ability” we must work hardest to cultivate in arithmetic and mathematics courses… [it] is even more important than knowledge and skill, because it enables to drive the necessary knowledge and skill” (Katagiri, 2006, p.5). Moreover, “mathematical thinking allows for (1) an understanding of the necessity of

using knowledge and skills (2) learning how to learn by oneself, and the attainment of the abilities required for independent learning" (Katagiri, 2006, p.6)
According to this philosophy, advancing mathematical thinking includes the development of:

- '**attitudes**'– intellectual predispositions towards doing mathematics and solving problems, including perspectives on what mathematics and mathematical activity are,
- '**contents**'– concepts, properties, interrelationships, and
- '**methods**'– inductive and deductive reasoning, analogical thinking, generalization, specialization, symbolization.

This rich view of what constitutes mathematical thinking drives teaching and was clearly reflected in all the lessons I saw and analyzed.

In the following, I concentrate on a particular aspect of the teaching practice: "teacher" actions, decision making, lesson crafting and the classroom setting which are aimed at the development, support and encouragement of sound and independent mathematics thinking. In other words, I describe what Japanese teachers actually do, how they do it, in order to engage students in thinking and learning while they are 'doing' mathematics, and how do teachers connect to students "en masse" in a fruitful and unconstrained way? (Lampert, 2001, p. 424).

Characteristics of the lessons

The following are some of the common characteristics of mathematical lessons in elementary school, which I found to be at the core of supporting mathematical thinking, and which are the result of purposeful planning and crafting by the teachers.

Coherence

All the lessons have a "story" – "A good story is highly organized; it has a beginning, a middle, and an end, and it follows a protagonist who meets challenges and resolves problems that arise along the way. Above all, a good story engages the reader's interest in a series of interconnected events, each of which is best understood in the context of the events that precede and follow it" (Stevenson & Stigler, 1992, p. 177). In other words, each lesson evolves and revolves around a central mathematical problem on which students work bringing in their common sense, their knowledge from outside mathematics, their findings from investigating the problem and their ongoing building on the ideas they produce in situ. The teacher leads students to apply knowledge and ideas that emerge during the lesson. The work has a unifying mathematical thread and sometimes the teacher creates an atmosphere of "suspense" around features of the problem, which fuels interest and maintains students engaged and active.
Coherent lessons that pursue central ideas around a meaningful and interesting problem are related to all the three components of mathematical thinking described above: attitudes, contents and methods. Whole lessons which pursue a central problem have the potential to nurture a view of mathematics as a discipline that tackles complex and relevant problems, which take some time to solve and which include attempts that fail and attempts that succeed, alternative approaches, discussion and exchange of ideas. The content involved

in solving a central problem goes beyond the presentation of a skill or a concept, the teacher involves the students in both conceptual understanding and in procedural activities, which are interwoven and are at the service of each other. In the process students propose methods of work and apply different ways of reasoning.

The following is an example of a lesson centered around one problem “the unfolding of the cylinder” (Arcavi, 2001). This is the second of three lessons (allotted by the curriculum) on the unfolding of solids. During this class, the problem is to design models of unfolded cylinders and then to assemble them in order to check that they indeed yield a cylinder. The goals are that students learn interactively (with concrete materials and with other students) about the structural components of the intervening two-dimensional figures, their relative positions, and, in certain cases, the importance of careful planning and measurements. In the process, students exercise their imagination, spatial visualization abilities, and creativity. The lesson opens with the teacher reminding the class of a previous lesson on the unfolding of a tetrahedron, and asks to think about the shape of an unfolded cylinder. After the class worked on the problem for a while, the teacher invites students to share their drawings on the blackboard. The first proposal is the classical: a rectangle and two tangent circles attached to its largest sides (prototypically, the largest sides are the horizontal). The teacher takes the opportunity to analyze the figure with the class, and to make sure students understand and agree on all the details. Thus, by asking several questions, simple but very important issues are raised and discussed, like:

- the two circles (the bases of the cylinder) should be of the same size,
- the two circles should be tangent to the a corresponding pair of parallel sides of the rectangle (and not secant to them),
- the length of the tangent sides should be equal to the circumference of the circles (students recall the number π and the formula for calculating the circumference),
- the length of the other two sides of the rectangle (to which the circles are not attached) are unconstrained (short sides and long sides will yield short or slender cylinders respectively),
- the points of tangency could be anywhere on each of the opposite sides of the rectangle.

Once these issues were discussed, the teacher encourages the class to produce alternative plane models for an unfolded cylinder. The class begins to propose other models including slicing and re-attaching parts of the bases, and many other creative designs, many of which will not fold into a cylinder. At a certain point, the teacher encourages the students to actually cut their designs, attempt to fold them into a cylinder and see if they succeed. In case of failure, students are encouraged to analyse the sources of their erroneous designs. By the end of the class, all the models are displayed.

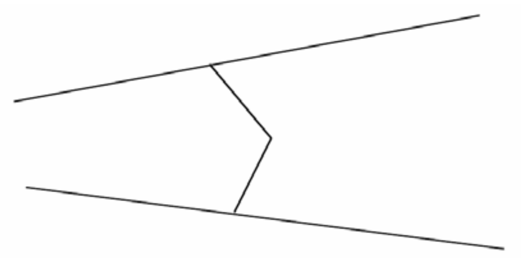
The “story” of this lesson has distinct parts: the discussion of the prototypical unfolding, the planning and design of alternatives, the practical work of assembling of the cylinders out of the models proposed, and the noticing and discussions of the failures. Coherence is not only a characteristic kept within the lesson (including the integration of visual reasoning with calculations, posing conjectures and checking them, analysing failed attempts and discussing each other’s solutions), but it is also related to the two other lessons this class has on the unfolding of solids.

Challenging problems

In most of the lessons I saw, there were instances in which I found myself solving mathematical aspects of the problem, as if I were a very engaged student participating in the class. I took this as an indication that the problems and discussions in these lessons are indeed mathematically interesting, challenging, and deep.

Consider, for example, a third grade lesson entitled "New ways of calculation" (for a detailed description see Arcavi 2007), in which the students are asked to calculate a series of multiplications of two numbers between 20 and 30 in which their unit digits add up to 10 (e.g. 23x27, 24x26, 25x25). As the lesson slowly unfolds, the teacher asked students to notice, record and communicate patterns (the way the exercises are handed in does not make the task of finding patterns a straightforward one), to propose an easy algorithm to perform these calculations, and to attempt to explain when and why it works. The new rule students discover and propose is that the result is 600 plus the result of the multiplication of the two digits, e.g. 23x27=600+3x7=621. Obviously, third grade students lack the algebraic tools to generalize and explain why the rule works, thus students work at the edge of their knowledge (or perhaps a little beyond that).

The following example is the central problem of the geometry class of the TIMSS video. "Replace the non-straight boundary dividing two pieces of land in the figure below



by a straight boundary line, while preserving the areas of the original pieces."

I have shared this problem with many knowledgeable mathematics teachers and they worked for a while before finding a solution, agreeing that this is a very difficult problem to be given to 8th graders. Certainly, this challenging problem was given within a coherent sequence of lessons and was supposed to be a non-trivial application of a property studied in a previous lesson: that all the triangles with the same base, and whose third vertex lies on a line parallel to that base, have the same area (the "constant area property").

The above problems are very different in nature, but they share some characteristics: they are not straightforward exercises, they require students to work at the edge of their knowledge, to explore, to discuss different approaches and to slowly device ways to make progress on the basis of mathematical content and different ways of reasoning.

The choice of problems like these implies that Japanese teachers feel very comfortable with their mathematical knowledge. But most importantly, by using these problems for an entire class, teachers enact their confidence that their students can and should engage with mathematical challenges and that they will be able to make progress.

Posing questions

The questions teachers pose to their students during the lessons and their questioning techniques are in consonance with the type of problems that teachers choose to be at the core of each lesson and at the service of solving them and learning from them. Many times the questions request explanations, arguments or counter-arguments. A salient characteristic of the lesson is to make sure that these explanations are fully understood by everybody, and one would expect the following line of questioning to attain this goal: requesting the student to repeat her explanation for the rest of the class, asking the class whether they understood it, asking who agrees or disagrees, and requesting other explanations. These are indeed part of the teacher's repertoire of questions. However, a technique which I found of great interest and importance is none of the above: after a student produces an explanation, an argument or proposes a conjecture the teacher asks the whole class who can explain such explanation or who can tell 'what is the thinking behind such explanation or proposal'. Such request nudges students to carefully learn to listen to each other, and before they can agree or disagree, to take the other's ideas and be able to replay and enact them as if they were theirs. Learning to listen to each other can be highly beneficial in developing the kinds of mathematical thinking the Japanese teachers are after in at least two different directions. Firstly, it may support the nurturing of empathic and caring relationships by conveying the message 'I take your idea and delve deep into it and its merits and sources'. Such a message is the precondition for the development of "academic civility" (Lampert, 2001, p. 431) in a classroom, within which all ideas are respected and valued and inspected mathematically. Secondly, by fully taking the other's perspective, one may be exposed to new ideas and forced to analyze them from within – on the one hand, this helps towards 'decentering' oneself and on the other hand, this may lead to re-inspect one's own knowledge, against the background of what was heard from others (Arcavi & Isoda, 2007). Thus, such a simple request from the teacher may be instrumental in supporting mathematical thinking.

There is another aspect to the questions teachers ask: the redefinition of the role of authority. In my observations, the teachers' authority is reflected in the decision about which task to focus on, which questions to pursue and when, how to distribute the right to speak and how to sequence the activities. The authority is not exerted in determining what is mathematically right or wrong, in this case the teacher deflects, as far as possible, such authority to the mathematics itself (Arcavi et al., 1998), placing a central role on the production of explanations and arguments to settle opposing results. This implies that erroneous answers are not immediately judged as incorrect, they have legitimate status until they are discussed against others. Building on students capacities to evaluate mathematical arguments and ideas places on them responsibility on their own learning and indeed supports the development of mathematical thinking.

Anticipation

Asking good open questions (such as requesting explanations, conjectures and proposals for strategies and ideas) constitutes, in more than one sense, a challenge for the teachers, because when students respond bringing in their proposals and ideas, teachers must do several things at once. Firstly, they have to perform an on site and very quick evaluation of the mathematical merit of the students' proposals, and this implies a solid mathematical background on the part of the teacher and a confidence to put it to use 'online'. Secondly, there must be an evaluation of the pedagogical possibilities that a student proposal affords and a decision regarding how to take advantage of it - this may imply changing the direction of the planned lesson and sometimes even some relinquishing of the control on the new directions the lesson may take. In my view, this is one the most difficult predicaments of the teaching profession. How do Japanese teachers cope with such situations? As far as I understand it, this issue is at the core lesson study: to study a lesson in depth and to implement it several times such that most students' reactions and proposals can be anticipated and only very few are new and surprising. Anticipated student reactions unload from the teacher the burden of on-site decision making. Furthermore, very fruitful student reactions which can contribute to the course of the lesson and which the teacher knows the students can produce them, can be stimulated and looked for at appropriate moments of the lesson.

Diversity

Much has been said about the ethnic (maybe also socio-economical) and cultural homogeneity of the Japanese society. Thus classroom realities are very different from those of the countries I know. Multiculturalism, multilingualism and social deprivation, which are pervasively reflected in classrooms in many Western countries are almost not known in Japan. However, there is another kind of diversity which is as present in Japanese classrooms as in their Western counterparts and which is no less of a challenge to teaching: children differ in their academic achievements and in their mastery of the subject. Japanese teachers cope with such diversity using different pedagogical approaches to a same topic, they have a proper pace and they harness all students' responses from the less to the more sophisticated (in that order) in the development of a lesson. However, no matter how competitive this society may be regarded by many, elementary schools do not track students and the teachers attend to all children in an impressively inclusive way. It is true that the respectful way in which Japanese people treat and address each other in ingrained in the culture and is the common and natural behavior: in contrast to what I have seen in many classrooms in other countries, I have not seen any Japanese teacher raising their voices or reprimanding students. Even when initial commotion was caused by the presence of visitors or by the mere playfulness of the children, the unrest quiets down by itself, mostly without the need of any intervention. Thus, the atmosphere teachers manage to induce in their classrooms is very propitious for thinking, working, asking any kinds of questions and freely expressing thoughts and ideas by everybody.

Pace
Some of my colleagues with whom I shared videos of Japanese lessons pointed to me that in many instances during the lesson they felt that the pace was slow. Interestingly, this is also the impression of many others (Stevenson & Stigler, p. 194). The slow pace of the lesson is related to its coherence (slowly building the "story" as described above) and to the teacher intention to be as inclusive as possible and to leave nobody behind. Moreover, this pace is a reflection of another deep belief: thinking takes time, ideas need to be mulled over, applied, discussed and approached from different directions, and if this is to be taken seriously there is no room for rushing.

Setting and devices
The 'architectural' setting of the classroom is traditional: lines of benches in rectangular rooms with a board at the front, simple teaching aids (such as magnetic manipulatives, paper cutting and the like). This setting does not prevent students from working with peers, come in groups to the board, and even moving around when the teacher thinks it is appropriate.
The blackboard plays a very central role in all classrooms, is not only a working space, but an organizational device, a thinking tool and a medium to record the flow of the lesson and its main ideas. Many times at the end of the class, if one looks at the board, it can tell the whole "story" of the lesson, especially displaying students' work and their differences of approaches.

Empathy
It is my impression that all students are affectively "contained" within a solid support net: teachers and parents work hard to closely follow up each of the students and attend to their needs as they arise. Teachers treat all their students with respect, they allow them to be boisterous and at times they even promote that. Some teachers also display their sense of humor, making the whole atmosphere of the class agreeable and supportive. Empathy seems to be yet another teaching strategy, which does not come at the expense of being intellectually demanding. Empathy seems to be characteristic of the way teachers address each other when discussing lesson plans and criticizing lessons. Teachers are used to expose their teaching to colleagues knowing that the analysis of their moves will be deep and thorough but very respectful and aimed at learning from each other.

The above characteristics are very different from each other. Some refer to a very deep pedagogical idea, some merely describe the physical setting, some present mathematical features, others refer to inter-human relationships. I would like to claim that maybe the uniqueness of the Japanese classrooms is due to the synergy of all these characteristics and to the professionalism with which of them each of them is treated.

Open questions

A first surprise regarding mathematics classrooms refers to something I did not see in them: computerized technologies. In spite of the many innovative proposals (and studies of their feasibility) about ways to introduce computerized and communication technologies both in the Japanese academia as well as in the academia of Western countries, I have not seen the use of computers in Japanese classrooms. In my view, the work with computerized technologies could fit in with the Japanese characterization of what mathematical thinking should be supported, and its availability in Japan is not an issue – thus, I wonder why it has not entered the classroom.

A second surprise refers to the shift in pedagogical practices that occur in secondary school, where most of the lessons consist of teacher lectures.

And finally, I wondered a lot about the existence, proliferation and success of "juku" schools (and possibly other out of school activities in mathematics), which are attended by a large number of students and are taken so much for granted by Japanese society at large, and which mostly emphasize drill and practice. This phenomenon can have several underlying reasons. For example, is it assumed and agreed that the time devoted by schools is insufficient, and drill should be learned elsewhere? Or, is it assumed that students need "extra practice"? Or, students should not have that much free time after school and their learning must be extended beyond the formal schooling? Or, students' full potential cannot be completely developed by the school only? Or, should students meet other teaching styles? Or….

References

Arcavi, A. (2007) “Exploring the unfolding of a cylinder” in Isoda, M., Stephens, M., Ohara, Y., & Miyakawa, T. (Eds.) *Japanese Lesson Study in Mathematics*, pp. 236-239. Singapore: World Scientific Publishing Co.

Arcavi, A. (2007) “New ways of calculation” in Isoda, M., Stephens, M., Ohara, Y., & Miyakawa, T. (Eds.) *Japanese Lesson Study in Mathematics*, pp. 240-243. Singapore: World Scientific Publishing Co.

Arcavi, A. & Isoda, M. (2007) “Learning to listen: from historical sources to classroom practice” *Educational Studies in Mathematics*. (Appeared online first, paper version will appear later in 2007, DOI 10.1007/s10649-006-9075-8)

Arcavi, A., Kessel, C., Meira, L. & Smith, J. (1998) “Teaching Mathematical Problem Solving: An analysis of an Emergent Classroom Community” *Research in Collegiate Mathematics Education*. *III*, 7, 1-70.

Arcavi, A. & Schoenfeld, A. H. (2006) “Using the unfamiliar to problematize the familiar” (in Portuguese) in M. Borba (Ed.) *Tendencias Internacionais em Formaçâo de Professores de Matemática* (*International Perspectives in Mathematics Teacher Education)*, pp. 87-111 (English version available).

Fernandez, C. & Yoshida, M. (2004) *Lesson Study. A Japanese Approach to Improving Mathematics Teaching and Learning*. Mahwah, NJ: LEA Inc., Publishers.

Katagiri, S. (2006) “Mathematical Thinking and How to Teach it” Lecture presented at APEC - Tsukuba International Conference. Full text available at http://www.criced.tsukuba.ac.jp/math/apec/apec2007/paper_pdf/Shigeo%20Katagiri.pdf

Lewin, K. (1952) *Field Theory in Social Science: Selected theoretical papers*. London: Tavistock.

Schoenfeld, A. H. (1998) “Toward a theory of teaching-in-context” *Issues in Education,* Volume 4, Number 1, pp. 1-94.

Stevenson, H. W. & Stigler, J. W. (1992) *The Learning Gap*. New York, NY: Simon & Schuster.

Stigler, J. W. & Hiebert, J. (1999) *The Teaching Gap*. New York, NY: The Free Press.

PLANNING A LESSON FOR STUDENTS TO DEVELOP MATHEMATICAL THINKING THROUGH PROBLEM SOLVING

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Teaching through problem solving has been emphasized in order to improve the teaching and learning of mathematics. However, it may not be easy for teachers to incorporate problem solving in their classrooms. An ideal way to incorporate problem solving is to plan a lesson and examine it through lesson study. This paper is intended to guide teachers in planning a lesson in which students will develop mathematical thinking through problem solving.

Developing mathematical thinking through problem solving

Teaching mathematics is for students to develop knowledge and skills that are mathematically important both for further study in mathematics and for use in applications in and outside of school is important for school mathematics. However, the objective of mathematics education is not only to enable students to acquire mathematical knowledge and skills but also to foster mathematical thinking. Mathematical thinking is crucial when students acquire and use mathematical knowledge and skills. In other words, students may have a difficult time acquiring and using knowledge and skills unless they have a sufficient ability to think mathematically.

In order to developing mathematical thinking, it is not enough for students simply to receive knowledge and skills by listening to teachers. Students need to actively engage in acquiring knowledge and skills, and to develop mathematical thinking through the process of mathematical activities. Thus students will be able to use these knowledge and skills effectively in their daily life as well as in their future carriers (Brown, 1994).

Based on the above assumption, it is suggested that teachers should provide students with opportunities to acquire knowledge and skills of mathematics through mathematical activities such as problem solving, reasoning and proof, communication, connection, and representation (National Council of Teachers of Mathematics, 2000). To implement such activity -based learning in mathematics classrooms, it is important for teachers to plan lessons that support students acquisition of the knowledge and skills by using mathematical thinking. Many teachers agree that teaching must emphasize the *process* of acquiring mathematics, But teachers often focus solely on teaching the contents to the students rather than providing students with opportunities to actually acquire the contents by using mathematical thinking. One of the reasons for teachers' hesitation to provide activities that cause students to develop mathematical thinking might be that the teachers have rarely experienced such lessons when they learned mathematics themselves. Moreover, planning lessons that focus more on students' learning process requires teachers to have more knowledge about their students, such as their thinking processes, in addition to having knowledge of the contents of mathematics (Simon & Tzur, 1999).

One of the ways to provide students with an opportunity to acquire not only knowledge and skills but also mathematical thinking is teaching mathematics through problem solving. Teaching mathematics through problem solving has been emphasized for decades, and many reform curriculum materials include problem solving as an integral part of learning mathematics (National Council of Teachers of Mathematics, 1980, 1989, 2000, 2006).

Problem solving in mathematics education is defined as "engaging in a task for which the solution method is not known in advance (National Council of Teachers of Mathematics, 2000)." This means that a problem suitable for problem solving is not necessarily a story problem or a problem in the real world. As long as a student does not know how to solve the problem, it can be a problem for problem solving for the student. In order word, even if a problem is presented as a real world story problem, it might not be a real problem for a student who already knows how to solve the problem. It is now called an exercise.

It is also important to note that teaching through problem solving is more than simply giving a task for students to solve a problem for which they have not learned the solution methods. Table 1. sho ws major differences between teaching of problem solving, which is a simplistic interpretation of the problem-solving approach, which can often be seen in traditional textbooks, and the teaching through problem solving, which is recommended by reform documents such as the NCTM Standards (2000).

Table 1. Problem-Solving Approach

Teaching of problem solving	Teaching through problem solving
• 'What is it? Problem Solving as an approach to develop problem-solving skills and strategies.	• 'What is it? Problem solving as a powerful approach for developing mathematical concepts and skills.
• How to incorporate it into a curriculum Usually the lessons based on this approach can be found at the end of chapters for developing problem-solving skills and strategies The lesson often end when each student comes up with a solution to the problem. (show and tell)	• How to incorporate it into a curriculum The lessons based on the approach can be found throughout the curriculum in order to develop mathematical concepts, skills, and procedures. The discussion for comparing students' different solutions is important for students to acquire new knowledge and understanding of mathematics.

Using lesson study to incorporate the idea of teaching through problem solving

Planning a lesson for lesson study is always challenging for teachers especially when teachers want to incorporate a new pedagogical idea that they have not experienced before. Lesson study is an ideal way to overcome such a challenging situation, because teachers can work together toward the same goal, which is to understand the new approach and to seek ways to incorporate it into each teacher's classroom (Takahashi & Yoshida, 2004).

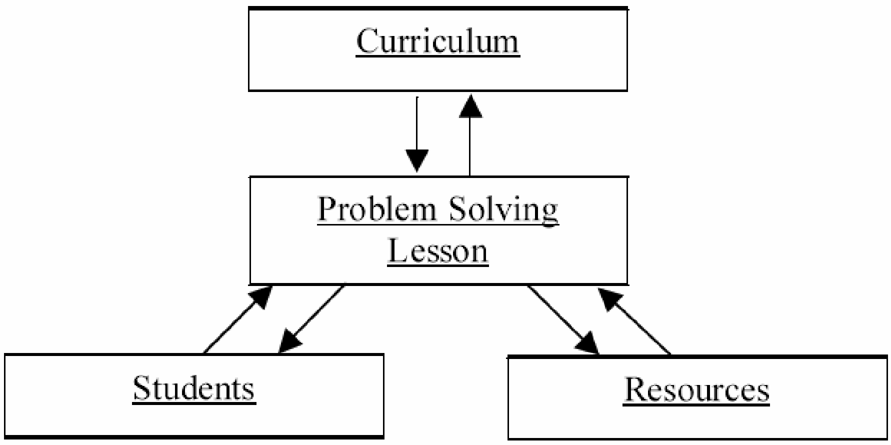
Although there are many different ways to plan a lesson for lesson study, it is often useful to examine each lesson carefully through the following three lenses; curriculum, students, and resources.

- Curriculum
  Any lessons that teachers prepare for their students must be purposeful and meaningful. Although there are many good problems for problem solving, it is important for teachers to identify what mathematics your students are expected to acquire through the lesson. This will become the goal of the lesson. In order to make the goal clear, teachers need to investigate the curriculum on which the students' mathematics learning is based – what they have already learned, what they are expected to learn now, and how their learning now will lead to their future learning. If the lesson is not fit into the curriculum well, it will not be helpful for students to accomplish the goal of the lesson.
- Students
  Knowing students is crucial when teachers plan a lesson. Especially for the lessons that are designed for students to acquire mathematical thinking, teachers need to know how the students might attempt to solve the problem. Without anticipating their students' approaches to the problem, teachers may not be able to plan how to lead the students to develop mathematical thinking by using their informal approaches to solving the problem. It is also important to anticipate students' typical misunderstandings so that teachers can be ready to support the students in overcoming their misunderstandings.
- Resources
  Choosing the best resources is also an important part of planning lessons. These resources include not only good problems in textbooks and other resource materials but also manipulatives, video, and interactive tools on the internet. Moreover, simply knowing what resources are available is not

enough. Teachers should know the potential benefits and drawbacks of each resource. For example, there are problems that are interesting and fun but that may not lead students to develop mathematical thinking at a particular time. When examing the lesson through this lens, teachers might want ask themselves if this is the best resource for students to reach the goal.

In Japanese, the above process of investigating the curriculum, students, and resources is called *Kyozai Kenkyu.* This investigation is important groundwork for planning lessons. The quality of each lesson will greatly rely upon the deepness of *Kyozai Kenkyu. Kyozai Kenkyu* becomes extremely important when teachers plan lessons to address new teaching agenda such as developing mathematical thinking through problem solving.

Figure 1. Planning a Problem Solving Lesson



Planning a lesson for students to develop Mathematical Thinking through Problem Solving

When begin to plan a problem-solving lesson, there are multiple entry points. For example teachers might want to begin *Kyozai Kenkyu* by carefully examining the curriculum to identify what mathematics the students are expected to acquire. Another entry point is to begin with examining students' work to identify what might be the area where students need to deepen their understanding in order to improve their mathematical ability. The third entry point is by examining resources to plan a problem-solving lesson. Although planning lessons in all three ways is encouraged for teachers, it is important to closely look at the lesson planning process by using the third entry point, resources, because many novice teachers take a wrong pass after they chose an attractive problem.

When teachers read teacher resource books and textbooks, and participate in professional development workshops and conferences, they often find an interesting and fun problem for students. Although these problems are useful resources, it is important to note that they are raw material and need to be prepared for a problem-solving lesson. It is not a good idea for teachers to bring those raw materials to classrooms to simply ask students to solve them or to show students how to solve problems. When teachers do not have a clear goal of the lesson, the lesson often become meaningless for the students.

At the APEC Tokyo/Sapporo symposia, Stacey (2006) used an interesting website, "Crystal Ball[1]", to illustrate the processes of mathematical thinking in the context of a problem solving lesson. If teachers are inspired by her talk and want to use the website to plan a problem-solving lesson for their own students to develop mathematical thinking, what should they do?

Investigating the problem

As Stacey (2006) describes, there are several ways to find out the trick. It is important for teachers to attempt several different approaches to discover mathematics behind the "Crystal Ball". In this particular case, teachers want to spend time to find the trick by themselves. Then, they should look at the same web page from students' viewpoints, asking themselves how could they find out the trick if they were a fifth grade student or a seventh grade student? This will lead teachers to investigate the problem through the student lens, although the investigation originally began through resource lens. After trying to figure out find out the trick in various ways, it might be a good idea to compare all the approaches for figuring out the trick to see how these approaches are related and how they are different. What mathematical knowledge and skills, and mathematical thinking are required for each approach?

Through this investigation, a group of teachers might be able to come up the following conclusion.

1 http://www.cyberglass.biz/customflash/ghostwhisperer/

In order to find out the trick, one of the approaches is to try several specific examples to find a pattern among the examples. Students typically use this inductive approach and find out that there might be mechanism behind the trick, but it is difficult to figure out why the pattern exists. Another approach is to investigate the process of calculations described in the “Crystal Ball” instruction in order to find out what calculations are actually carried out to get the symbol that you need to imagine. This deductive approach demands that students write, interpret, and use mathematical expressions to investigate the trick, then find out why the crystal ball gives you the same symbol no matter what two digit numbers are chosen. During this investigation, students will be using their previous learning of the properties of the basic operations, the notion of place value, and the use of symbols in mathematical expressions to see the generalized pattern.

Investigating the problem through other lenses

The next step toward planning a lesson by using the “Crystal Ball” might be to narrow down what mathematics teachers expect students to acquire through this problem solving. From previous investigation, teachers agree that most students should be able to try at least a couple of specific cases to draw a conclusion that there might be a trick behind the website. Moreover, some of the students might be able to find that the procedure that the “Crystal Ball” gives always produces a number that is a multiple of nine. It is, however, expected that many students might not be able to figure out why the procedure always gives a number that is multiple of nine, because it requires students to manipulate mathematical expressions.

The above process describes how teachers can investigate the problem through the lens of "students". The next step might be the investigation through the "curriculum" lens. In the *Curriculum focal points for pre -kindergarten through grade 8 mathematics: a quest for coherence* (National Council of Teachers of Mathematics, 2006), one of the focal points in the middle school is to write, interpret, and use mathematical expressions and equations to solve problems. It is expected that students become able to

1) write mathematical expressions and equations that correspond to given situations,
2) evaluate expressions, and
3) use expressions and formulas to solve problems.

One of the challenges for the students is to write mathematical expressions that correspond to a given situation. Sometimes students may be reluctant to write mathematical expressions because they often try to find the answer by simply carrying out calculations and cannot see the merits of writing mathematical expressions. In order to overcome students’ reluctance to write mathematical expressions, therefore, it is important that they learn how writing mathematical expressions that can help them to solve problems.

Form that discussion, the goal of this problem-solving lesson might be to provide students an opportunity to learn inductive reasoning by writing, interpreting, and using mathematical expressions.

Designing the flow of the lesson

After going through the ground work, *Kyozai Kenkyu,* the group of teachers should move toward actually discussing how to pose the problem, and what questions a teacher can ask the students toward acquiring mathematical knowledge and skills. There are several types of lesson plans for lesson study. One important component that most Japanese lesson plans share is the section called "anticipated students' responses." The quality of the section of student anticipated responses relies heavily on the richness of *Kyozai Kenkyu.* Moreover, this section usually contributes greatly to the quality of the discussion that a teacher will be leading after students present their various solutions.

It is often a good idea for teachers to prepare answers for the following questions to develop a short sketch of the lesson.

- Purpose of the problem solving (goal of the lesson)

  What mathematics, beside developing problem solving skills, would you teach by using this situation?

- Questioning

  How would you pose the problem?
  What question(s) would you ask of your students for them to learn mathematics?

- Beyond show and tell

  Anticipate students' responses to your questions, including misunderstandings, to facilitate discussion.
  Briefly describe how you would facilitate discussion.

The Appendix shows an example of the lesson plan for the problem-solving lesson using the "Crystal Ball."

Conclusion

Planning lessons for lesson study demands that teachers spend time and effort. Although it is time consuming, once teachers experience this process with their colleagues, they start seeing their everyday lessons differently. It is not easy for teachers to change their teaching practice in a short time. However, it will be a great step toward addressing mathematical thinking in their everyday lessons.

It will be also a powerful experience for teachers to observe an actual classroom based on the lesson plan that the group planned together. Since the mathematical thinking can be observed mostly in the process of students' problem solving and dialogues among students, the entire process of lesson study is expected to improve mathematical thinking.

1 http://www.cyberglass.biz/customflash/ghostwhisperer/

References

Brown, A. (1994). The Advance of learning. *Educational Researcher, 23(8),* 4-12. National Council of Teachers of Mathematics, (2006). *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence.* National Council of Teachers of Mathematics. (1980). *An agenda for action: Recommendations for school mathematics of the 1980s.*

Reston, Virginia: National Council of Teachers of Mathematics. National Council of Teachers of Mathematics. (1989). *Curriculm and evaluation standards for school mathematics*. Reston, Virginia: National Council of

Teachers of Mathematics.National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston,

Virginia: National Council of Teachers of Mathematics.National Council of Teachers of Mathematics. (2006). *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence.*

Simon, M. A., & Tzur, R. (1999). Explicating the teacher's perspective from the researchers' perspectives: Generativing account of mathematics teachers' practice. *Journal for Research in Mathematics Education, 30(3),* 252-264.

Stacey, K. (2006). What is Mathematical Thinking and Why is It Important? *Progress report of the APEC project: "Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (II) - Lesson Study focusing on Mathematical Thinking -".*

Takahashi, A., & Yoshida, M. (2004). How Can We Start Lesson Study?: Ideas for establishing lesson study communities. *Teaching Children Mathematics, Volume 10, Number 9.,* pp.436-443.

8TH GRADE MATHEMATICS LESSON PLAN

April 26, 2007
Las Cruces, NM
Instructor: Akihiko Takahashi

1. Title of the lesson: The Secret of The Crystal Ball
2. Goal of the lesson:
   1. To deepen students' understanding of the properties of the basic operations and place value by writing, interpreting, and using mathematical expressions through problem solving
   2. To help students become good problem solvers by
      i. encouraging them to use their prior knowledge to examine a problem situation in order to develop their ability to use logical reasoning to make conjectures, and
      ii. encouraging them to examine and justify the conjectures presented by their peers in order to find a solution to the problem.
   3. Provide opportunities for students to recognize the importance of working with their peers in order to deepen their understanding of mathematics

3. Instruction of the Lessons

In the Curriculum focal points for pre-kindergarten through grade 8 mathematics: a quest for coherence (National Council of Teachers of Mathematics Inc. Reston VA., 2006), one of the
focal points in the middle school is to write, interpret, and use mathematical expressions and equations to solve problems. It is expected that students become able to:

1) write mathematical expressions and equations that correspond to given situations,
2) evaluate expressions, and
3) use expressions and formulas to solve problems.

One of the challenges for the students is to write mathematical expressions that correspond to a given situation. Sometimes students may be reluctant to write mathematical expressions because they often try to find the answer by simply carrying out calculations and cannot see the merits of writing mathematical expressions. In order to overcome students' reluctance to write mathematical expressions, therefore, it is important that they learn how writing mathematical expressions can help them to solve problems.

When designing such problem-solving lesson, it is important to keep in mind that solving a problem is a process for providing an opportunity for students to appreciate that writing, interpreting, and using mathematical expressions. Therefore, the flow of the lesson should not solely focus on finding the correct answer, but also the process of solving the problem.

This lesson is designed for students' to understand how writing, interpreting, and using mathematical expressions help them analyze the problem situation and empower them to solve a problem.

The problem for this lesson to figure out the mechanism behind a trick named "Crystal Ball" from the website of a popular TV program, Ghost Whisperer (http://www.cbs.com/primetime/ghost _whisperer/crystal b all.shtml). The website is based on a popular math trick and use Flash, multimedia authoring program for web applications, to make it interactive and engaging. The procedures that described on the website is

> *Chose any two digit number, add together both digits and then subtract the total from your original number. When you have the final number look it up on the chart and find the relevant symbol. Concentrate on the symbol and when you have it clearly in your mind click on the Ghost Whisperer crystal ball and it will show you the symbol you are thinking of*

In order to find out the trick, one of the approaches is to try several specific examples to find a pattern among the examples. Students typically use this inductive approach and find out that there might be mechanism behind the trick but it is difficult to figure out why the pattern exists. Another approach is to investigate the process of calculations described in the "Crystal Ball" instruction in order to find out what calculations are actually carried out to get the symbol that you need to imagine. This deductive approach demands that students write, interpret, and use mathematical expressions to investigate the trick, then find out why the crystal ball always gives you the same symbol no matter what two digit numbers are chosen. During this investigation, students will be using their previous learning of the properties of the basic operations, the notion of place value, and the use of symbols in mathematical expressions to see the generalized pattern.

4) Flow of the Lesson

Learning Activities, Teacher's Questions and Expected Students' Reactions	Teacher's Support	Points of Evaluation
1. Introduction to the Problem By experiencing the "Crystal Ball" on the internet, students will become familiar with the site. 1. Chose any two digit number, 2. Add together both digits, 3. Subtract the total from your original number 4. When you have the final number look it up on the chart and find the relevant symbol. 5. Concentrate on the symbol and when you have it clearly in your mind 6. Click on the crystal ball to see the symbol	Ask a couple of volunteer students to try the website so that all the students understand the procedures described on the webpage. Help students to see the website always gives you the relevant symbol.	Do students understand the procedure? Do students see what is happening on the website?
2. Posing the problem By asking the following question, engage students to find the trick behind the "Crystal Ball" webpage. With which opinion do you agree? a. It is just a coincident and there is nothing special in the "Crystal Ball" webpage. b. There might be a trick behind the "Crystal Ball". c. The "Crystal Ball" webpage actually reads your mind. Let's find the trick behind the "Crystal Ball" webpage!	Each student will be working with his/her partner to find a trick by using their prior knowledge. Provide students with worksheets to keep their work for the whole class discussion.	Do students see there must be a trick behind the "Crystal Ball" webpage

3. Problem Solving Working with a partner, students try to find the trick behind the "Crystal Ball" webpage. Anticipated students' responses: a. Try a couple of specific examples to notice that the relevant symbol might be always the same but do not know why these symbols are the same. b. By examining several specific examples, he/she realizes that the final number will always be a multiple of nine, and the symbols on the chart that correspond to multiple of nine are all the same. However, he/she does not know why the final number will always be a multiple of nine. c. Write, interpret, and use mathematical expressions to investigate the trick - $a\ b$ as a chosen two digit number - The value of $a\ b$ is $10a + b$ - Write a mathematical expression to express the procedure $(10a + b)(a + b)$ $= 10a + b a b$ $= 10 a a + b b$ $= 9a$ Therefore the final number will always be a multiple of nine	Encourage students to try at least a couple of specific examples. Help students understand that methods (a) and (b) may not be able to answer the question why all the final numbers give you the same symbol. Encourage students to investigate the process of calculations described in the instructions to "Crystal Ball" in order to find out what calculations are actually carried out to get the symbol that you need to imagine.	Do students try at least a couple of specific examples to notice that the relevant symbol from your calculation might always be the same.
4. Discussing Students' Solutions (1) Ask students to explain their solutions to the other students in the class. (2) Facilitate students' discussion about their solutions, then lead them to understand that writing, interpreting, and using mathematical expressions helped them understand the trick behind the "Crystal Ball" webpage.	Write students' solutions and ideas on the blackboard in order to help students understand the discussion.	Can students explain their solutions to their peers? Can students examine and justify the solutions presented by their peers?
5. Summing up (1) Using the writing on the blackboard, review what students learned through the lesson. (2) Ask students to write a journal entry about what they learned through this lesson.	Encourage students to use the writing on the board as a reference when they write the journal entry.	

Reference

National Council of Teachers of Mathematics Inc. Reston VA. (2006). *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence.*

http://www.cbs.com/primetime/ghost whisperer/crystal ball .shtml

Board writing Plan for the "Crystal Ball"

The Crystal Ball

1. Chose any two digit number,
2. Add together both digits,
3. Subtract the total from your original number
4. When you have the final number look it up on the chart and find the relevant symbol
5. Concentrate on the symbol and when you have it clearly in your mind click on the crystal ball to see the symbol

What is happening on the website?
Use the worksheet to figure out

Case 1: 56
5+6=11
56-11=45 Symbol A

Case 2: 78
7+8=15
78-15=63 Symbol A

Students' approach A

Students' approach B

Students' approach C

The Crystal Ball always give you the same symbol no matter what two digit numbers are chosen because

1. the final number that the procedure give by the Crystal Ball always be a multiple of 9,
2. the symbols on the chart that correspond to multiple of 9 are all the same. (with two exceptions, 90 and 99)

SETTING LESSON STUDY WITHIN A LONG-TERM FRAMEWORK OF LEARNING

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Lesson Study is a format to build and analyse classroom teaching where teachers and researchers combine to design lessons, predict how the lessons might be expected to develop, then carry out the lessons with a group of observers bringing multiple perspectives on what actually happened during the lesson. This article considers how a lesson, or group of lessons, observed as part of a lesson study may be placed in a long-term framework of learning, focusing on the essential objective of improving the long-term learning of every individual in classroom teaching.

INTRODUCTION

This paper began as a result of a participation in a lesson study conference (Tokyo & Sapporo, December 2006) in which four lessons were studied as part of an APEC (Asian and Pacific Economic Community) study to share ideas in teaching and learning mathematics to improve the learning of mathematics throughout the communities. It included the observation of four classes (here given in order of grade, rather than order of presentation):

Placing Plates (Grade 2)
December 2nd 2006, University of Tsukuba Elementary School
- Takao Seiyama

Multiplication Algorithm (Grade 3)
December 5th 2006, Sapporo City Maruyama Elementary School
- Hideyuki Muramoto

Area of a Circle (Grade 5)
December 2nd 2006, Universty of Tsukuba Elementary School
- Yasuhiro Hosomizu

Thinking Systematically (Grade 6)
December 6th 2006, Sapporo City Hokuto Elementary School
- Atsutomo Morii

The objective of this paper is to set these classes within a long -term framework of development outlined in Tokyo at the conference (Tall, 2006), which sets the growth of individual children within a broader framework of mathematical development. Long-term the development of individual children depends not only on the experiences of the lesson, but in the experiences of the children prior to the lesson and how experiences 'met-before' have been integrated into their current knowledge framework.

In general, it is clear that lesson study makes a genuine attempt:

to design a sequence of lessons according to well-considered objectives;
to predict what may happen in a lesson;

to have a group of observers bring multiple perspectives to what happened, without prejudice; and ultimately

to improve the teaching of mathematics for all.

Lesson study is based on a wide range of communal sharing of objectives. At the meeting I was impressed by one essential fact voiced by Patsy Wang -Iverson:

> The top eight countries in the most recent TIMMS studies shared a single charactistic, that they had a smaller number of topics studied each year.

Success comes from focusing on the most generative ideas, not from covering detail again and again. This suggests to me that we need to seek the generative ideas that are at the root of more powerful learning.

For many individuals, mathematics is *complicated* and it gets more complicated as new ideas are encountered. For a few others, who seem to grasp the essence of the ideas, the *complexity* of mathematics is fitted together in a way that makes it essentially *simple* way. My head of department at Warwick University in the sixties, Sir Christopher Zeeman noted perceptively:

> "Technical skill is a mastery of complexity, while creativity is a mastery of simplicity" (Zeeman, 1977)

This leads to the fundamental question:

> How can we help *each and every child* find this simplicity, in a way that works, *for them?*

Lesson study focuses on the *whole class activity*. Yet within any class each child brings differing levels of knowledge into that class, related not only to what they have experienced before, but how they have made connections between the ideas and how they have found their own level of simplicity in being able to think about what they know.

To see simplicity in the complication of detail requires the making of connections between ideas and focusing on essentials in such a way that these simple essentials become generating principles for the whole structure.

In my APEC presentation in Tokyo (Tall 2006), I sought this simplicity in the way that we humans naturally develop mathematical ideas supported by the shared experiences of previous generations. I presented a framework with three distinct worlds of mathematical development, two of which dominate development in school and the third evolves to be the formal framework of mathematical research. The two encountered in school are based on (conceptual) embodiment and (proceptual) symbolism. I described these technical terms in more detail in Tall (2006) and in a range of other recent papers on my website (www.davidtall.com/papers).

Essentially, conceptual embodiment is based on human perception and reflection. It is a way of interacting with the physical world and perceiving the properties of objects and, through thought experiments, to see the essence of these properties and begin to verbalise them and organize them into coherent structures such as Euclidean geometry. Proceptual symbolism arises first from our *actions* on objects (such as counting, combining, taking away etc) that are symbolized as concepts (such as number) and developed into symbolic structures of calculation and symbolic manipulation through various stages of arithmetic, algebra, symbolic calculus, and so on. Here symbols such as 4+3, $x^2 + 2x + 1$, ! sin x dx all dually represent processes to be carried out (addition, evaluation, integration, etc) and the related concepts that are constructed (sum, expression, integral, etc). Such symbols also may be represented in different ways, for instance 4+3 is the same as 3+4 or even '1 less than 4+4' which is '1 less than 8' which is 7. This flexible use of symbols to represent different *pro*cesses for giving the same underlying con*cept* is called a *procept.*

These two worlds of (conceptual) embodiment and (proceptual) symbolism develop in parallel throughout school mathematics and provide a long-term framework for the development of mathematical ideas throughout school and on to university, where the

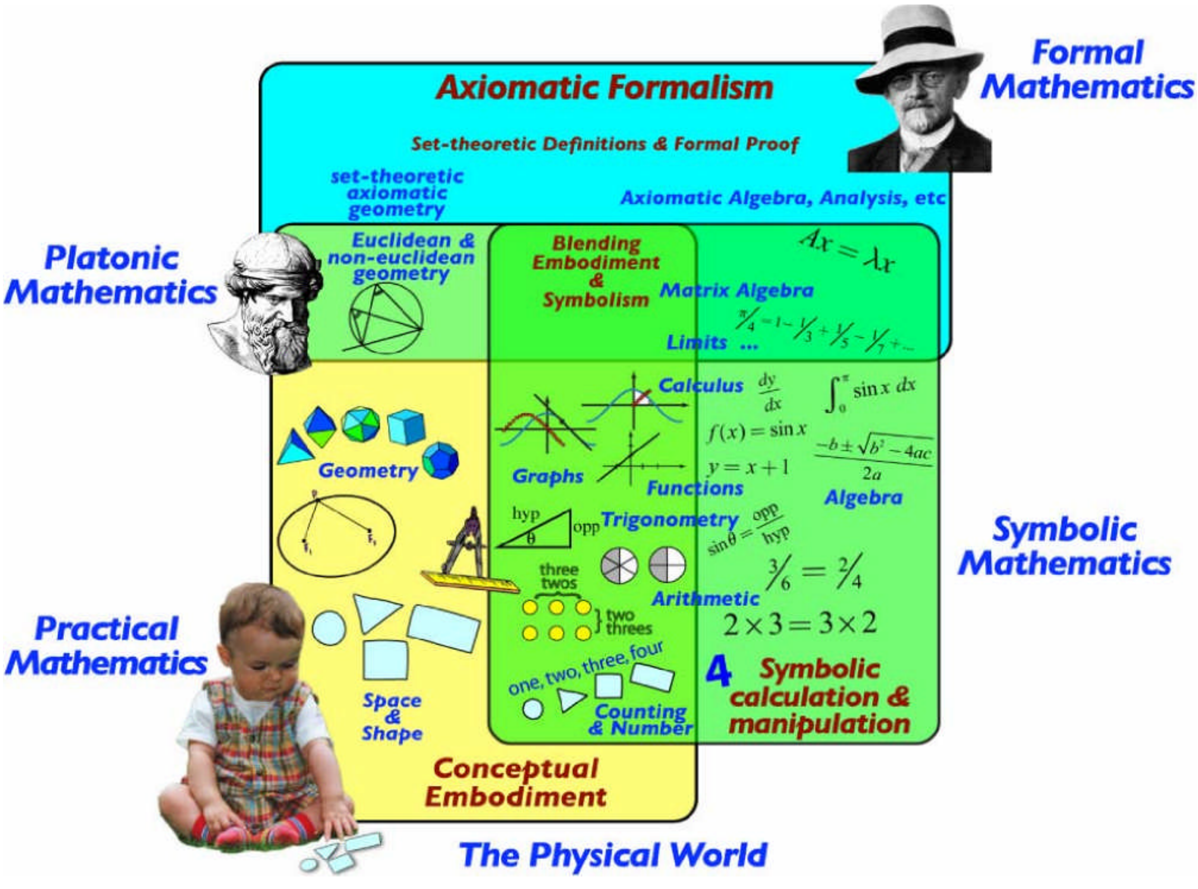


Figure 1. The three mental worlds of (conceptual) embodiment, (proceptual) symbolism and (axiomatic) formalism

focus changes to the formal world of set-theoretic definition and formal proof.

In figure 1 we see an outline of the huge *complication* of school mathematics. On the left is the development of conceptual embodiment from practical mathematics of physical shapes to the platonic methods of Euclidean geometry. In parallel, there is a development of symbolic mathematics through arithmetic, algebra, and so on, with the two blending as embodiment is symbolized or symbolism is embodied.

The long-term development begins with the child's perceptions and actions on the physical world. In figure 1 the child is playing with a collection of objects: a circle, a triangle, a square, and a rectangle. The child has two distinct options, one to focus on his or her *perception* of each object, seeing and feeling their separate properties, the other is through *action* on the objects, say by counting them: one, two, three, four.

The focus on perception, with vision assisted by touch and other senses to play with the objects to discover their properties, leads to a growing sense of space and shape, developing through the use of physical tools—ruler, compass, drawing pins, thread—to enable the child to explore geometric ideas in two and three dimensions, and on to the mental construction of a perfect platonic world of Euclidean geometry. The focus on the essential qualities of points having location but no size, straight lines having no width but arbitrary extensions and on to figures made up using these qualities leads the human mind to construct mental entities with these essential properties. Platonism is a natural long-term construction of the enquiring human mind.

Meanwhile, the focus on action, through counting, leads eventually to the concept of number and the properties of arithmetic that benefit from blending embodiment and symbolism, for example, 'seeing' that 2 x 3= 3 ! 2 by visualizing 2 rows of 3 objects being the same as 3 columns of 2 objects. Long-term there is a development of successive number systems, fractions, rationals, decimals, infinite decimals, real numbers, complex numbers. (What seems to the experienced mathematician as a steady extension of number systems is, for the growing child, a succession of changes of meaning which need to be addressed in teaching. We return to this later.)

The symbolic world develops through whole number arithmetic, fractions, decimals, algebra, functions, symbolic calculus, and so on, which are given an embodied meaning through the number-line, Cartesian coordinates, graphs, visual calculus, with aspects of the embodied world such as trigonometry being realized in symbolic form. In the latter stages of secondary schooling, the learner will meet more sophisticated concepts, such as symbolic matrix algebra and the introduction of the limit concept, again represented in both embodied and symbolic form.

The fundamental change to the formal mathematics of Hilbert leads to an axiomatic formalism based on set-theoretic definitions and formal proof, including axiomatic geometry, axiomatic algebra, analysis, topology, etc.

Cognitive development works in different ways in embodiment, symbolism and formalism (Figure 2). In the embodied world, the child is relating and operating with perceived objects (both specific and generic), verbalizing properties and shifting from practical mathematics to the platonic mathematics of axioms, definitions and proofs.

In the symbolic world, development begins with actions that are symbolized and coordinated for calculation and manipulation in successively more sophisticated contexts. The shift to the axiomatic formal world is signified by the switch from concepts that arise from perceptions of, and actions on, objects in the physical world to the verbalizing of axiomatic properties to define formal structures whose further properties are deduced through mathematical proof.

Focusing on the framework appropriate to school mathematics, we find the main structure consists of two parallel tracks, in embodiment and symbolism, each building on previous experience (met-befores), with

> *embodiment* developing through perception, description, construction, definition, deduction and Euclidean proof after the broad style suggested by van Hiele;
>
> *symbolism* developing through increasingly sophisticated compression of procedures into procepts as thinkable contexts operating in successively broader contexts.

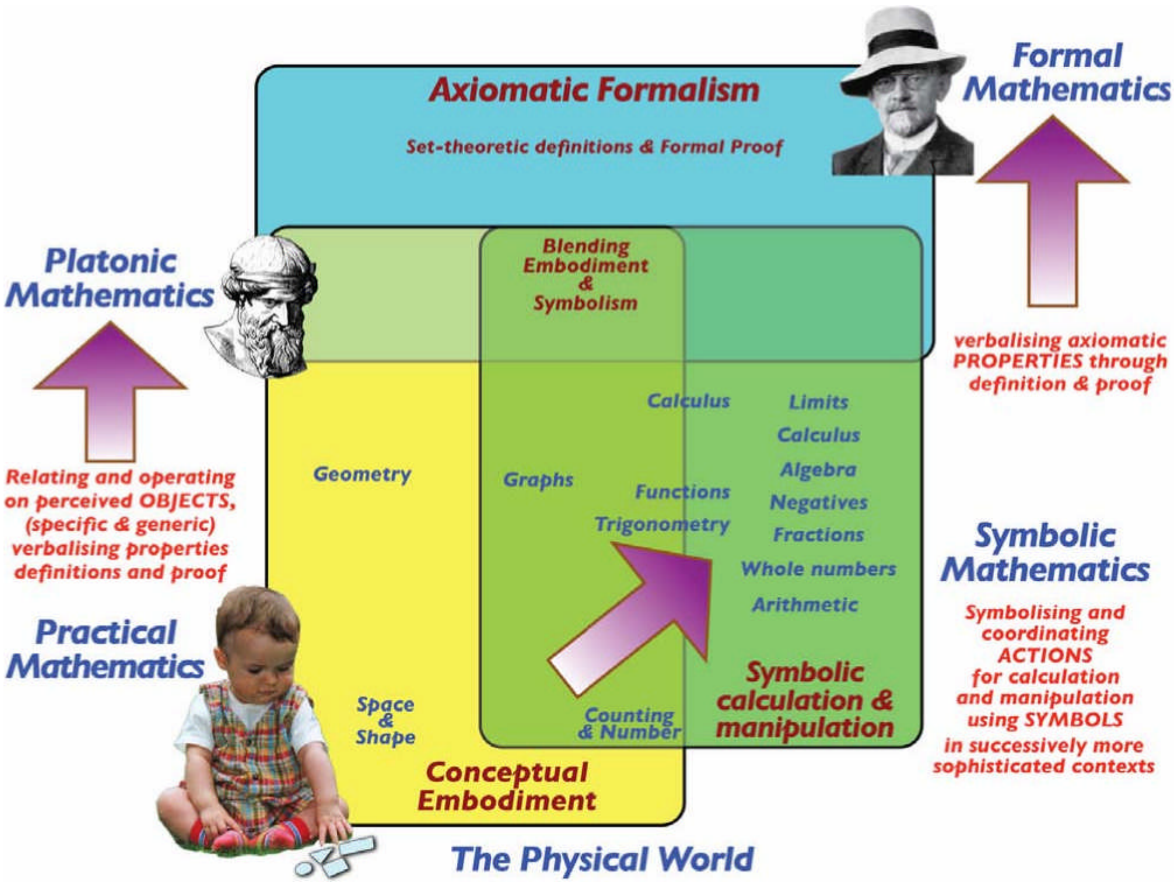


Figure 2: long-term developments in the three worlds

These two developments are fundamentally different. On the one hand, embodiment gives a global overall picture of a situation while symbolism begins with coordinating actions, practicing sequences of actions one after another to build up a procedure, perhaps refining this to give different procedures that are more efficient or more effective, using symbolism to record the actions as thinkable concepts. The problem here is that the many different procedures can, for some, seem highly complicated and so the teacher faces the problem of reducing the complexity, perhaps by concentrating on a single procedure to show the pupils what to do, without becoming too involved in the apparent complications. Procedures, however, occur *in time* and become routinized so that the learner can *perform* them, but is less able to *think about* them. (Figure 3.)

As an example, consider the teaching of long-multiplication. First children need to learn their tables for single digit multiplication from 0 ! 0 to 9 ! 9. They also need to have insight into place value and decimal notation.

The method used by Hideyuki Muramoto in the lesson study at Sapporo City Maruyama Elementary School on December 6, 2006 can be analysed in terms of an initial embodiment representing 3 rows of 23. Here the learner can *see* the full set of counters: the problem is how to *calculate* the total. The embodiment can be broken down in various ways, separating each row into subsets appropriate to be able to compute the total. In the previous lesson the students had already considered 3 rows of 20 and had broken this into various sub-combinations, breaking each row into 10+10 or 5+5+5+5, or even 9+9+2, or 9+2+9. Now the problem related to breaking

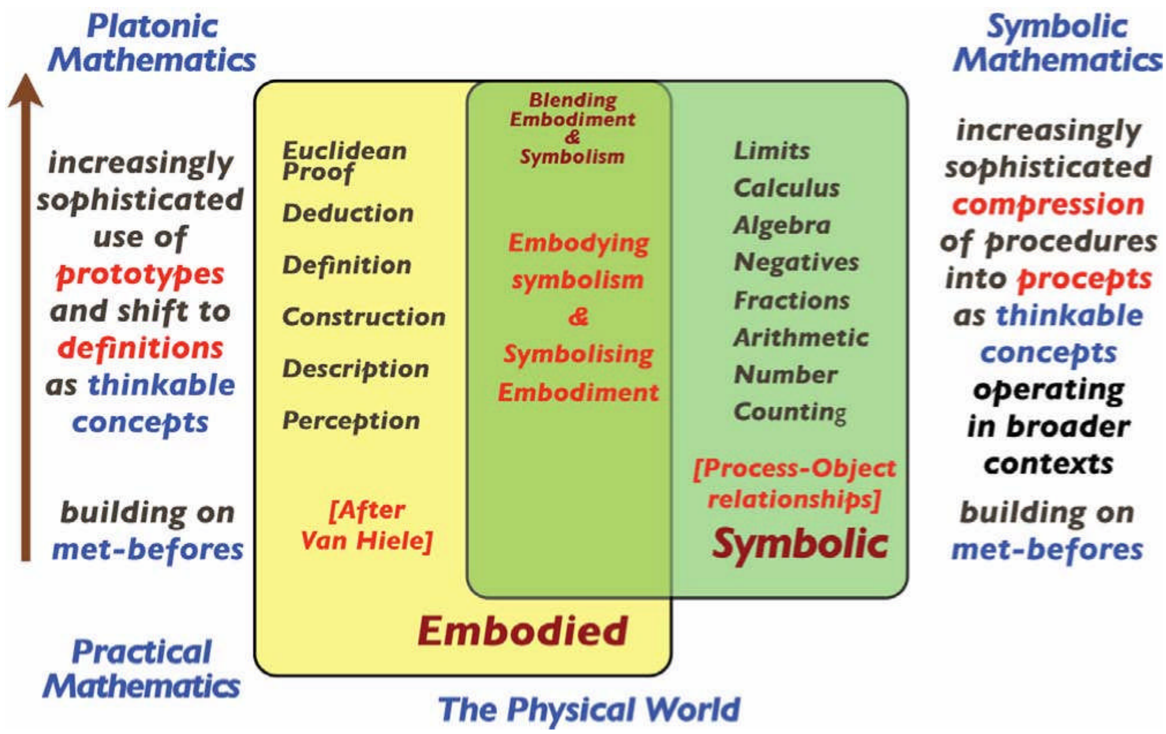


Figure 3: Developmental framework through embodiment and symbolism

23 into sub-combinations, suggested possibilities included 10+10+3 and 9+5+9 (but not 5+5+5+5). Three lots of 10+10+3 gives 30+30+9, which easily gives 60+9, which is 69. Three lots of 9+5+9 is more difficult requiring the sum 27+15+27. Here we have two different procedures giving the same result, 69, and the most productive way forward is to break the number 23 into tens and units and multiplying each separately by 3.

In this analysis, the embodiment gives the *meaning* of the calculation of a single digit times a double digit number, while the various distinct sub-combinations give different ways of *calculation*, from which the sub-combination as tens and units is clearly the simplest and the most efficient.

The approach has a general format:

1. *Embody the problem* (here the product 23 ! 3);
2. *Find several different ways of calculation* (here 23 ! 3 is three lots of 10+10+3 or three lots of 9+9+5) *where the embodiment gives meaning to symbolism;*
3. See *flexibility*, that all of these are the same;
4. See the *standard algorithm is the most efficient.*

Thus embodiment gives meaning while symbolism enables compression to an efficient symbolic algorithm.

It may be that not all the children in the class will be able to cope with the different procedures (for instance, one would expect the suggestion 9+5+9 to come from a more able child and the computation would not be easy for some). Thus, the dynamic of the whole class may not be shared by all individuals. The more successful may see the different ways of computing the result as different procedures with the same

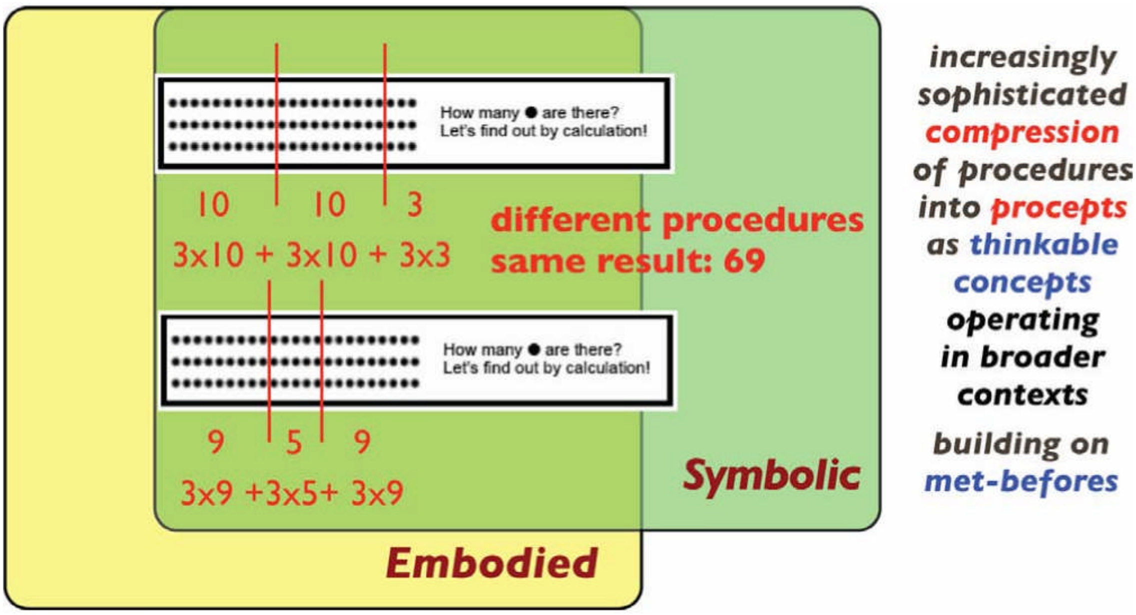


Figure 4: multi-digit arithmetic from embodiment to symbolism

effect, and meaningfully see that the standard algorithm is just one of many that is chosen because it is efficient and simple. They may sense that it is not appropriate to use a more complicated method like 3 times 9+9+5 and not even desire to carry it through without this compromising their insight that different procedure s can give the same result. Meanwhile, those who are less fluent in their tables may feel insecure and seek an easy method to cope that is less complicated. A single procedure may have its attractions, showing *how to do it,* without the complication of *why it works.* It may have attractions to the teacher to teach the method by rote as this may have short-term success without extra complication.

In this way, the same lesson may be seen very differently by different participants, at one extreme, a great insight into the meaning and construction of the standard algorithm within a rich conceptual framework, at another extreme, a great deal of complication and a desire to cope by seeking a procedure *that works* rather than a situation which is too complicated to understand. This bifurcation is what Gray & Tall (1994) called the *proceptual divide* between those who seek to maintain procedures that work at the time rather than flexible methods that require many meaningful connections in a broader knowledge structure.

BLENDING KNOWLEDGE STRUCTURES IN THE BRAIN

In addition to this combination of embodiment and symbolism to give meaning to number concepts and operations, there are subtle features of successive number systems that cause additional problems. A mathematician may see successive numbers systems such as:

Whole Numbers

Fractions

Rational Numbers

Positive and Negative numbers

Real Numbers consisting of rationals and irrationals

as a growing extension of the number system. They can all be marked on an (embodied) number line and the child should be able to *see* how each one is extended to the next. However, for the learner, each extension has subtle aspects which can cause significant problems. We all know of the difficulty of introducing the concept of fraction and of the problem of multiplying negative numbers. There are subtle difficulties between counting and measuring:

Counting 1, 2, 3, ... has successive numbers, each with a next number and no numbers in between. Multiplying these numbers gives a bigger result ... etc.

Measuring numbers are continuous without a ‘next’ number and have fractions between. Multiplying can give a smaller result.

Elsewhere (e.g. Tall, 2007), I use the idea of *conceptual blending* from Fauconnier & Turner (2003) to shed light onto the cognitive strengths and difficulties of long -term learning in mathematics. Fauconnier and Turner share the distinction of being the first cognitive scientists to integrate the fundamental ideas of *compression* and *blending* of knowledge into a single framework. In considering how students learn long-term, this suggests we need to be aware not only what experiences students have had before, but how they compress this experience into thinkable concepts and how different knowledge structures are blended together to produce new knowledge.

USING A LONG-TERM FRAMEWORK OF EMBODIMENT AND SYMBOLISM IN LESSON STUDY

Putting together the ideas of growth in elementary mathematics discussed here and in the earlier paper (Tall, 2006), we find that the parallel development of embodiment and symbolism suggests:

> Embodiment gives human meaning as prototypes, developing verbal description, definition, deduction.
>
> Symbolism is based initially on human action, leading to symbol use, either through procedural learning or through conceptual compression to flexible procept.
>
> Experiences build met-befores in the individual mind that are used later to interpret new situations.
>
> Different experiences may be blended together, requiring a study of what learners bring to a new learning experience.

Tall (2006) also observed:

> Embodiments may work well in one context but become increasingly complex; flexible symbolism may extend more easily.

This means that successful students may show a long-term tendency to shift to symbolism to work in a way that is both more powerful and (for them) more simple.

In our earlier discussions in Tokyo, great emphasis was made not only on meaningful learning of mathematical concepts and techniques, but also on *problem-solving* in new contexts. Learning new concepts can be approached in a problem-solving manner. My own view is that learners must take responsibility for their own learning, once they have the maturity to do so, which includes developing their own methods for solving problems. I also believe that teachers have a duty, as mentors, to help focus students on methods that are powerful and have long-term value.

In studying lessons, therefore, we need some objectives to consider. There are so many theories in the literature, from Bruner's (1966) analysis into enactive iconic and symbolic, Fischbein's (1987) categorization into intuitive, algorithmic and formal, the Pirie-Kieren theory (1994) with its ideas of 'making' and 'having' images and successive levels of operation, Dreyfus and colleagues RBC theory (Recognising, Building-With, Consolidating), theories of problem-solving (Schoenfeld 1985, Mason *et al.* 1982) and so on. With such a wealth of ideas to choose from and build on (and build with), I will hear focus on three simple ideas that are important. You may choose different ones, but in the long run, it is important for those studying lessons to have principles with which they are working and a fundamental framework for each lesson study. I suggest the need in long-term development to focus on three aspects:

Building thinkable concepts in (*meaningful)* knowledge structures;

Using knowledge structures in *routine* and *problem* situations (where 'routine' includes practising for fluency);

Proving knowledge structures (*as required in context).*

I would see these three aspects being applied *before, during* and *after* each lesson.

BEFORE: What is the purpose of the lesson

(e.g. **Building** new constructs, **Using** *known routines* or *problem-solving*, **Proving** in some sense) and what concepts may the learners have in mind that may be used in the lesson? (*met-befores, blends, routines, problem-solving techniques)*

DURING: How do learners use their knowledge structures during the lesson to make sense of it? (*met-befores, blends, routines, problem-solving techniques)*

AFTER: What knowledge structures are developing that may be of value in the future? (*met-befores, blends, routines, problem-solving techniques)*

LESSONS STUDIES

Four classes were videoed during our previous meeting in Japan, December, 2006.

Placing Plates (Grade 2)
December 2nd 2006, Universty of Tsukuba Elementary School
- Takao Seiyama;

Multiplication Algorithm (Grade 3)
December 5th 2006, Sapporo City Maruyama Elementary School
- Hideyuki Muramoto;

Area of a Circle (Grade 5)
December 2nd 2006, University of Tsukuba Elementary School
- Yasuhiro Hosomizu;

Thinking Systematically (Grade 6)
December 6th 2006, Sapporo City Hokuto Elementary School
- Atsutomo Morii.

My purpose is to focus on the role of these lessons in long-term learning, and to consider how the long-term development of each and every student may be affected by the lesson within the framework suggested above.

There is already a great deal of evidence of the use of broad principles in the planning of the lessons which are formulated in the lesson plans. Taking a few quotes at random we find:

> The goal of the Mathematics Group at Maruyama is to develop students ability to use what they learned before to solve problems in the new learning situations by making connections.
>
> In addition, we want to provide 3rd grade students with experiences in mathematics that enable them to use why they learned before to give problems in new learning situations by making connections.
>
> Through teaching mathematics, I would like my students to develop 'secure ability' for finding problems on their own, studying by themselves, thinking, making decisions, and executing those decisions. Moreover, I would like to help my students like mathematics as well as enjoy thinking.
>
> In order for students to find better ideas to solve the problem, it is important for the students to have an opportunity to feel that they really want to do so.
>
> Starting in April (beginning of the school year), I taught the students to look at something from a particular point of view such as 'faster, easier, and accurate' when they think about something or when they compare something.
>
> If you think about the method that uses the table form this point of view, students might notice that "it is accurate but it takes a long time to figure out: or "it is accurate but it is complicated."
>
> In order to solve a problem in a short time and with less complexity, it is important for the students to notice that calculation using a math sentence in necessary.

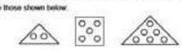
Each of these shows a genuine desire for students to make connections, to rely on themselves for making decisions and to seek more powerful ways of thinking with less complexity. The videos of the classes themselves show high interaction between the students, and with the teacher, carefully orchestrated by the teacher to bring out essential ideas in the lesson.

We now briefly look at each lesson in turn, to see how it fits with a long-term development blending embodiment and symbolism, what aspects of Building, Using, and Proving arise as an explicit focus of attention, before, during, and after the lesson. In particular, we need to look deeper at how individual children respond to the lesson in ways that may be appropriate for their long-term development of powerful mathematical thinking.

In the pages which follow, I reproduce overheads from my presentation that look at each of the lessons to see where it fits in the overall plan of building ideas from a blend of embodiment and symbolism to build use and prove powerful mathematical concepts. This is, in no way, intended to be a once-and-for-all analysis. It is offered as a preliminary analysis for those developing lesson study to initiate discussion on how to implement the techniques of lesson study within a long-term framework that focuses on improving the learning of mathematics for each and every student.

A Long-Term Learning Framework for Lesson Study
Placing Plates (Grade 2)
December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama

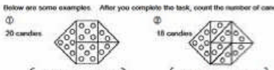
There are candies placed on small plates that are shaped like triangles and a quadrilateral, just like those shown below.



One of the tasks of this lesson is to make a large hexagonal plate by fitting together small plates like those shown above. Rules for making a large plate are as follows:

> You must fit together the small plates and make a shape that matches the large plate exactly.

Below are some examples. After you complete the task, count the number of candies.



$\left(5 \times 2 = 10 \quad 2 \times 2 = 4 \quad 10 + 6 + 4 = 20\right)$ $\quad$ $\left[2 \times 6 = 12 \quad 12 + 6 = 18\right]$

Students will notice the difference between the number of candies on the various small plates by using multiplication which the students learned before to find out the number of candies. After students present various solutions to this problem, I would like to expand the lesson by paying attention to students' awareness of the problems involved.

Using ideas in a non-routine, *problem-solving* activity.

[**Proving** by physical embodied experiment]

Met-before: shapes, simple arithmetic

Activity: how to think flexibly in a specific problem situation

Long-term: flexible thinking with specified rules, encouraging a problem-solving attitude in an idiosyncratic problem.

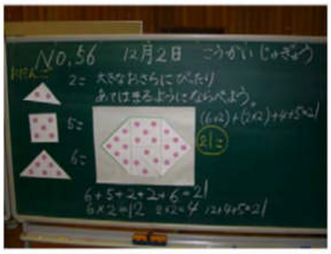
A Long-Term Learning Framework for Lesson Study
Placing Plates (Grade 2)
December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama

Experimenting with the problem

A Long-Term Learning Framework for Lesson Study
Placing Plates (Grade 2)
December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama

Sharing data

A Long-Term Learning Framework for Lesson Study
Placing Plates (Grade 2)
December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama



Organizing data

A Long-Term Learning Framework for Lesson Study
Placing Plates (Grade 2)
December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama



An enjoyable well-planned activity allowing a wide range of levels of performance.

What is the contribution to future development?

Some practice of arithmetic
Flexible problem-solving
(e.g. finding all possible combinations)
Some idiosyncratic data eg squares can have 5 or 6 candies on them.
Questions:
What is the *important long-term role* of this lesson that the children should focus on?
What do individual children learn from this experience in the long-term?

A Long-Term Learning Framework for Lesson Study
Placing Plates (Grade 2)
December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama



There are two objectives for this lesson. The first one is to foster students' geometric sense through composition of geometric shapes. And the second one is to foster students' ability to think logically and understand mathematical expressions by asking them to think about the composition of geometric shapes and their matching mathematical expressions.

3. Goals of the Unit

- ○ To foster student understanding of triangles and quadrilaterals through concrete manipulative activities.
- ○ To enrich the basic learning experiences of students by composing and drawing triangles and quadrilaterals.

4. Instructional Plan (Total: 6 periods)

Phase 1: Meaning of triangles and quadrilaterals —— 2 periods
Phase 2: Composition and construction of triangles and quadrilaterals —— 2 periods
Phase 3: Summary and practice —— 1 period

Questions:
What is the *important long-term role* of this lesson that the children should focus on?
What do individual children learn from this experience in the long-term?

A Long-Term Learning Framework for Lesson Study

Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

How many ● are there?
Let's find out by calculation!

A Long-Term Learning Framework for Lesson Study

Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

How many ● are there?
Let's find out by calculation!

Building ideas in a flexible manner.

Met-before: single-digit multiplication subdividing a problem into smaller problems

Activity: constructing different ways of calculating 3 times 23

Long-term: flexible thinking about multiplication, revealing the standard algorithm as the most efficient.

A Long-Term Learning Framework for Lesson Study

Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

Experimenting with the problem

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Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

Discussing ideas

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Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

Explaining to the teacher

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Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

Displaying different solutions

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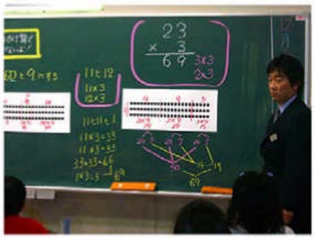
Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto



Comparing solutions for efficiency

A Long-Term Learning Framework for Lesson Study

Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

A well-organised lesson in a sequence designed to give a flexible insight into multiplication.

What is the contribution to future development? Different children brought different met-befores. Some struggled with the arithmetic, some already knew the long-multiplication algorithm.

Questions:
What is the *important long-term role* of this lesson that the children should focus on? What do individual children learn from this experience in the long-term?

A Long-Term Learning Framework for Lesson Study

Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

Goals of the Unit:

- Lessons that enable students to consciously think about the connection between what they learned before and what they are learning now
- Lessons in which students learn from each other and that help them consciously think about their own solution processes
- An evaluation method that helps foster students' logical thinking abilities.
- Unit plan
- This lesson (goals, process of lesson)

Questions:
What is the *important long-term role* of this lesson that the children should focus on? What do individual children learn from this experience in the long-term?

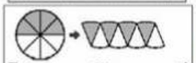
A Long-Term Learning Framework for Lesson Study

Area of a Circle (Grade 5)

December 5, 2006, University of Tsukuba Elementary School - Yasuhiro Hosomizu

1. Present the problem

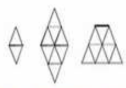
Come up with ways to find the area of the circle by using the sectors that are made by segmenting the circle into eight



The area of parallelogram=base×height

2. Think about different ways to rearrange the shape so that other formulas for finding the areas of basic shapes can be used

Rearrange the shape and find different formulas to find the area



dividing into 8 and 16 pieces

Building ideas using embodiment in a flexible manner.

Met-before:
area unchanged when parts are moved without overlap. Possible problem with curved edges.

Activity: making up areas from a subdivided circle

Long-term: meaningful understanding of the area of a circle.

A Long-Term Learning Framework for Lesson Study

Area of a Circle (Grade 5)

December 5, 2006, University of Tsukuba Elementary School - Yasuhiro Hosomizu

Thinking about the problem

A Long-Term Learning Framework for Lesson Study

Area of a Circle (Grade 5)

December 5, 2006, University of Tsukuba Elementary School - Yasuhiro Hosomizu

Making up solutions

A Long-Term Learning Framework for Lesson Study

Area of a Circle (Grade 5)

December 5, 2006, University of Tsukuba Elementary School - Yasuhiro Hosomizu

Explaining

A Long-Term Learning Framework for Lesson Study

Area of a Circle (Grade 5)

December 5, 2006, University of Tsukuba Elementary School - Yasuhiro Hosomizu

Summarizing

A Long-Term Learning Framework for Lesson Study

Area of a Circle (Grade 5)

December 5, 2006, University of Tsukuba Elementary School - Yasuhiro Hosomizu

A well-organised lesson in a sequence designed to give a flexible insight into various ways of seeing the area of a circle.

What is the contribution to future development?
Designed to give meaning to the area of a circle. Questions remain about the curved edges in the area which could be seen in terms of 'local straightness' in calculus.

Questions:
What is the *important long-term role* of this lesson that the children should focus on?
What do individual children learn from this experience in the long-term?

A Long-Term Learning Framework for Lesson Study

Area of a Circle (Grade 5)

December 5, 2006, University of Tsukuba Elementary School - Yasuhiro Hosomizu

3. Unit plan (Circle, total 10 lesson periods)
1st section Circle and regular polygons (2 lesson periods)
2nd section Length of circumference (3 lesson periods)
3rd section Area of circle (3 lesson periods) today's lesson 2/3
4th section Summary and application (2 lesson periods)
4. Goal of the lesson
1) Goal
Students will be able to come up with ways to find the area of a circle by rearranging the shape of the circle so that they can use previously learned formulas, and be able to derive the formula for finding the area of a circle.

Questions:
What is the *important long-term role* of this lesson that the children should focus on?
What do individual children learn from this experience in the long-term?

A Long-Term Learning Framework for Lesson Study

Thinking Systematically (Grade 6)

December 6, 2006, Sapporo City Hokuto Elementary School - Atsutomo Morii

4. Process of the lesson

Students' activities and thinking process	Teacher's support
We bought pencils and ballpoint pens and the total number of items were 10 and the price was 460 yen. The price of each pencil was 40 yen and the ballpoint pen was 70 yen. How many pencils and how many ballpoint pens did we buy? If we calculate it… If we make a table…	○ Listening to the students' muttering (or voices) and pick up the idea to use a table to solve this problem. Then ask the students to fill in the table on the worksheet.

# of pencils	0	1	2	3	4	5	6	7	8	9	10
# of ballpoint pens	10	9	8	7	6	5	4	3	2	1	0
Total price (yen)	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

The table in the textbook shows the number of pencils and ballpoint pens from 1 to 9, but in this lesson I decided to use the number from 0 to 10. This is decision relates to my hope for a certain kind of mathematical thinking that I want my students to acquire.

More sophisticated solutions:

If we buy only ballpoint pens the total price would be 700 yen.
(700 – 460) ÷ 30 = 8
Thus, the answer is 8 pencils and 2 ballpoint pens.

If we buy only pencils the total price would be 400 yen.
(460 – 400) ÷ 30 = 2
Thus, the answer is 2 ballpoint pens and 8 pencils.

Building ideas relatin
2 variables using tables.

Using *problem-solving* to use the data systematically

Met-before:
Previous experience of relationships & tables.

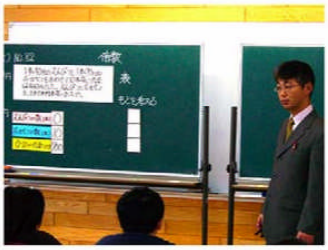
Activity: more subtle solutions possible, but main focus on tables.

Long-term: to realise tables are systematic, but tedious, to generat the need for a more powerful way of expressing and solving the problem.

A Long-Term Learning Framework for Lesson Study

Thinking Systematically (Grade 6)

December 6, 2006, Sapporo City Hokuto Elementary School - Atsutomo Morii



Starting a table with zero

A Long-Term Learning Framework for Lesson Study

Thinking Systematically (Grade 6)

December 6, 2006, Sapporo City Hokuto Elementary School - Atsutomo Morii

Building examples of data as columns

A Long-Term Learning Framework for Lesson Study

Thinking Systematically (Grade 6)

December 6, 2006, Sapporo City Hokuto Elementary School - Atsutomo Morii

Organising the complete table

A Long-Term Learning Framework for Lesson Study

Thinking Systematically (Grade 6)

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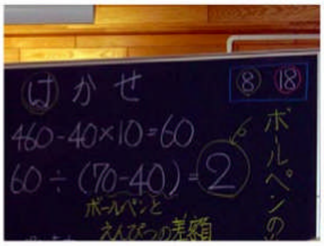

A more sophisticated solution

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Intention of this lesson and mathematical thinking would like to foster among the students

During 4th grade, students learned how two quantities change in the unit called "investigating changes in quantities." In the 4th grade, they also learned how to express the relationship between two quantities using tables and math sentences. In addition, the students had experience reading the changes of the quantities and their characteristics.

In the 5th grade, based on their experience in 4th grade, students learned to solve problems by finding the relationship between two quantities and their regularity using tables.

The aim of this lesson is to use knowledge from prior grade levels to solve problems using tables that have more items. This lesson is included in the mathematics textbook. This lesson is not included as a part of a unit but it is set up as individual lesson. Title of the next unit is "proportional relationships." In the unit, students will construct tables, finding regularity, and expressing the relationship using math sentences. I believe this lesson is included here to help students prepare to learn about proportional relationships.

In this lesson, I anticipate that the students might solve this problem by coming up with an appropriate value and then calculating or by constructing a table. I believe that constructing a table is not a difficult task for the students because of their prior learning experiences. Moreover, I believe that many of the students will use a table to solve the problem.

The table in the textbook shows the number of pencils and ballpoint pens from 1 to 9, but in this lesson I decided to use the number from 0 to 10. This is decision relates to my hope for a certain kind of mathematical thinking that I want my students to acquire.

I would like to focus on a kind of mathematical thinking, i.e. hypothetical thinking.
Something like, "If it is ...... then ......"

By changing the quantities of the items in the problem on their own, the students can come up with better solution methods. In order to do that I think it is important for the students to see an extreme case in the table such as " I bought 10 items of one kind and 0 items of the other kind."

Questions:

What is the *important long-term role* of this lesson that the children should focus on?

What do individual children learn from this experience in the long-term?

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Summary

Four lessons:

- 2 building new concepts: multiplication, area of circle,
- 2 using problem-solving: plates, thinking systematically.

All based on shared working, structured by the teacher; and all fitting into the long-term framework specified here.

The two building new concepts are part of a long-term development (as is thinking systematically).

The problem-solving activities have general principles of self-construction, sharing, making sense for one's self, etc, but do they have any problem-solving strategies?

There are general comments about desired learning.

What about analysis of *individual* development?

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Comment

After formulating a theory of conceptual blending and possible misconceptions, this rarely features in the analysis!

In part the particular lessons do not reveal much about these issues.

Nor have we collected much *individual* data from the lessons.

The lessons are all planned for large classes with the focus on what is to be learned in a commendably flexible way, rather than on the range of possible individual learning that may reveal significant differences in performance.

In Britain, attention is turning to the needs of 'pupils at risk' who need extra support and to the 'gifted and talented' who need extra challenges.

> É for pupils at risk of falling behind, early intervention and special support to help them catch up. This is already underway with the 'Every Child a Reader' programme for literacy, which is now being matched with the 'Every Child Counts' initiative for numeracy, alongside one-to-one tuition for up to another 600,000 children.Gordon Brown, The Guardian, May 15, 2007

However, it is not a linear race, with some 'falling behind' and others 'racing ahead'. It is also a question of different kinds of learning and different ways of coping.

Assuming our major purpose is to improve the long-term learning of mathematics for each and every one of our children, I suggest that there is a need for lesson study to be placed in a long-term framework to design and monitor the long-term development of individuals, to gain insight not only what needs to be learnt and how, but also why some develop flexible, powerful mathematical thinking and others have serious difficulty.

The framework offered is based on the different styles of cognitive growth in embodiment and symbolism over the long -term, and the way in which different individuals build on mental structures based on ideas met-before.

REFERENCES

Bruner, J. S. (1966). *Towards a Theory of Instruction,* New York: Norton.

Fauconnier & Turner (2002). *The way we think: Conceptual Blending and the Mind's Hidden Complexities.* New York: Basic Books.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach.* Dordrecht, Holland: Kluwer.

Gray, E. & Tall, D. O.: 1994, Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *The Journal for Research in Mathematics Education,* 26(2), 115–141.

Mason J., Burton L. & Stacey K. (1982). Thinking Mathematically, Addison-Wesley. Pirie, S., & Kieran, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics,* 26, 165 –190.

Schoenfeld, A. (1985). *Mathematical Problem Solving.* New York: Academic Press.

Tall, D. O. (2004). Thinking through three worlds of mathematics, *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education,* Bergen, Norway, 4, 28 1–288.

Tall, D. O. (2006). Encouraging Mathematical Thinking that has both power and simplicity. Plenary presented at the APEC -Tsukuba International Conference, December 3–7, 2006, at the JICA Institute for International Cooperation (Ichigaya, Tokyo).

Tall, D. O. (2007). Embodiment, Symbolism, Argumentation and Proof, Keynote pres ented at the Conference on Reading, Writing and Argumentation at National Changhua Normal University, Taiwan, May 2007.

Zeeman, E. C. (1977). *Catastrophe Theory: Selected Papers, 1972 -1977.* Addison-Wesley.