TEACHERS' MATHEMATICAL THINKING

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Introduction

In my presentation at the Tokyo 2006 APEC symposium I demonstrated that mathematical thinking is important in three ways.

- Mathematical thinking is an important goal of schooling.
- Mathematical thinking is important as a way of learning mathematics.
- Mathematical thinking is important for teaching mathematics.

I spent most of that presentation discussing the first two dots points, and only discussed the third point with one example. In this presentation, I will discuss the third point in more depth. I ended my presentation at the last symposium with these comments:

"For those us who enjoy mathematical thinking, I believe it is productive to see teaching mathematics as another instance of solving problems with mathematics. This places the emphasis not on the static knowledge used in the lesson as above but on a process account of teaching. In order to use mathematics to solve a problem in any area of application, whether it is about money or physics or sport or engineering, mathematics must be used in combination with understanding from the area of application. In the case of teaching mathematics, the solver has to bring together expertise in both mathematics and in general pedagogy, and combine these two domains of knowledge together to solve the problem, whether it be to analyse subject matter, to create a plan for a good lesson, or on a minute-by-minute basis to respond to students in a mathematically productive way. If teachers are to encourage mathematical thinking in students, then they need to engage in mathematical thinking throughout the lesson themselves."

The first announcement for the December 2006 Tokyo APEC conference states that a teacher requires mathematical thinking for analysing subject matter (p. 4), planning lessons for a specified aim (p. 4) and anticipating students' responses (p. 5).These are indeed key places where mathematical thinking is required. However, in this section, I concentrate on the mathematical thinking that is needed on a minute by minute basis in the process of conducting a good mathematics lesson. Mathematical thinking is not just in planning lessons and curricula; it makes a difference to every minute of the lesson. In this analysis, I aim to illustrate how strong and quick mathematical thinking provides the teacher with many possible courses of action. The course of the lesson, though, is then determined by how the teacher weighs up the possibilities which he or she sees. The mathematical possibilities are considered along with knowledge of students' mathematical understandings and needs and with pragmatic factors (eg those associated with keeping the lesson on track), and a choice is made. These decisions determine the course of a lesson.

We now examine the mathematics used by two teachers when their classes tackle the 'spinners game'. After this, I also report on experiences when the problem was adapted and used in a primary teacher education class.

Irene's lesson on the Spinners Game

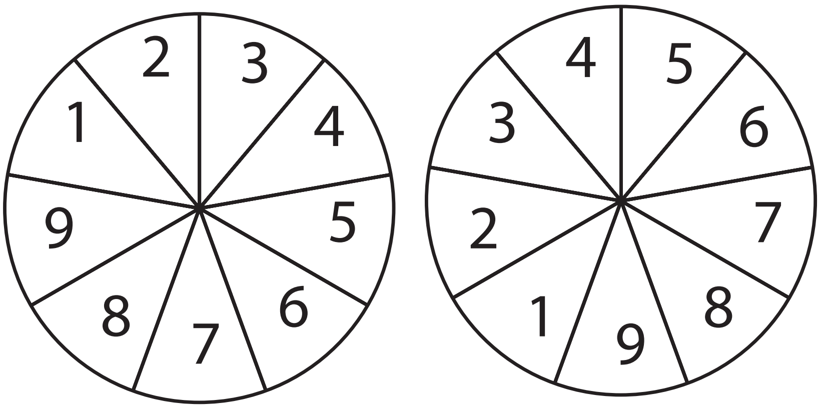

Figure 1. Equipment for the spinners game

The spinners game was first discussed in Chick and Baker (2005) and is based on their classroom observations and interviews with the teachers. This account of two classroom uses is reproduced with adaption from Chick (2007) with permission, and additional points relevant to this presentation have been inserted. Irene, an experienced teacher, and Greg, who was in only his second year of teaching, were Grade 5 teachers in the same school. They had chosen to use a spinner game suggested in a teacher resource book (Feely, 2003). The spinner game used two spinners divided into nine equal sectors, labelled with the numbers 1-9 (see Figure 1). The worksheet instructed students to spin both spinners, and add the resulting two numbers together. If the sum was odd, player 1 won a point, whereas player 2 won a point if the sum was even. The first player to 10 points was deemed the winner. Students were further instructed to play the game a few times to "see what happens", and then decide if the game is fair, who has a better chance of winning, and why (Feely, 2003, p. 173). The teacher instructions (Feely, 2003, p. 116) included a brief suggestion about focusing on how many combinations of numbers add to make even and odd numbers but did not provide any additional direction.

This game can offer worthwhile learning opportunities associated with sample space, fairness, long-term probability, likelihood, and reasoning about sums of odd and even numbers. The significant issue here, especially in the absence of explicit guidance from the resource book is *how* these learning opportunities can be brought out. Although it is not written in the teachers' resource book, the spinners game has an interesting twist. Analysis of the sample space shows that the chances of Player 2 (even) winning a point is 41/81 compared to 40/81 for Player 1 (odd). Player 2 is therefore theoretically more likely to win, however this miniscule difference in likelihood implies that the game's theoretical unfairness will not be evident when playing "first to ten points". We cannot tell whether the authors of the resource book chose this narrow difference deliberately or accidentally. Our interest here is in the teachers' mathematical thinking as they implemented the activity in the classroom.

Irene started the spinners' game late in a lesson. Most students had played the game for a few minutes before Irene began a short class discussion. She asked the class if they thought it was a fair game. Discussion ensued, as students posed various ideas without any of them being completely resolved. For instance, someone noted that fairness requires that players play by the rules of the game. Most of the arguments about fairness were associated with the number of odds and evens, both in terms of the individual numbers on the spinners (there are 5 odds and only 4 evens on each spinner) and in terms of the sums. One student neatly articulated an erroneous parity argument, that since "odd + odd = even and even + even = even but odd + even = odd, therefore Player 2 has two out of three chances to win". Irene said she was not convinced about the "two out of three", but she agreed the game was unfair.

The student's presentation of this argument, which Irene suspects is not valid, requires her to make a decision as to whether it should be pursued, or passed over quickly in favour of something else. She might, for example, have presented (or sought from a student) another erroneous argument along the same lines but which takes into account the fact that odd + even and even + odd occur in different ways: "odd + odd = even and even + even = even but odd + even = odd *and even + odd = odd*, therefore *both* players have two out of *four* chances to win". Presenting this argument would have emphasised that the different orders are important, but the new argument has the same failing as the first argument. It does not take into account that there are different numbers of odd and even numbers. Instead Irene might have decided to highlight just this failing of the student's argument, showing, for example that odd + odd is more likely than even + even. The several possibilities for responding to the argument as well as the possibility of simply passing over it quickly, as she chose to do, must be identified and evaluated in just a few seconds as the classroom discussion proceeds.

Good decisions would seem to be enhanced when teachers see the mathematical possibilities quickly and evaluate them from a mathematical point of view (what important mathematical principles/processes/strategies/attitudes would the students learn from this). However, decision making also needs to be informed by knowledge of how the students will respond, and by attention to practical aspects of the lesson, including the time available. For Irene, the necessity to finish the spinners game in the few remaining minutes of the lesson might have been the over-riding consideration.

Irene then allowed one of the students to present his argument. At the start of the whole class discussion this student had indicated that he had not played the game at all but had "mathsed it" instead, and at that time Irene made a deliberate decision to delay the details of his contribution until the other students had had their say. He proceeded to explain that he had counted up all the possibilities, to get 38 even totals and 35 odd totals. Although this was actually incorrect, Irene seemed to believe that he was right and continued by pointing out that this meant that “it’s [the game is] not *terribly* weighted but it *is slightly* weighted to the evens”. Irene then asked the class if their results bore this out, and highlighted that although the game was biased toward Player 2 this did not mean that Player 2 would always win.

As suggested earlier, the spinner game provides the opportunity to examine sample space, likelihood, and fairness. Given the impact of time constraints on Irene’s lesson, sample space was not covered well, although she believed that the student who had “mathsed it” had considered all the possibilities. This highlights a contrast between her knowledge of his capabilities and of the details of the content with which he was engaged. On the other hand, her content knowledge was sufficient for her to recognise the significance of the small difference between the number of odd and even outcomes and its impact on fairness. Irene led a good discussion of the meaning of fairness and the magnitude of the bias, and its consequence for the ‘first to ten’ aspect. Given the short time available to end the lesson, it may have been a wise decision to ignore the errors in the student’s sample space and go on to what Irene probably saw as the main point: that the difference in likelihood is very small, and that even if there is a bias students would not have been able to reliably detect it in the ‘first to ten’ game.

In considering Irene’s lesson, we see that its path is determined by many small decision points: who to call on next, whether to check the student’s list of outcomes or simply believe him because he is a good student, whether to pursue the errors in the parity argument etc. These decisions are influenced by factors relating to the mathematics (as perceived during the flow of the lesson by the teacher), factors relating to the students’ current knowledge and factors relating to the pragmatic conduct of the lesson (e.g. how much time is left). This is illustrated in Figure 2.

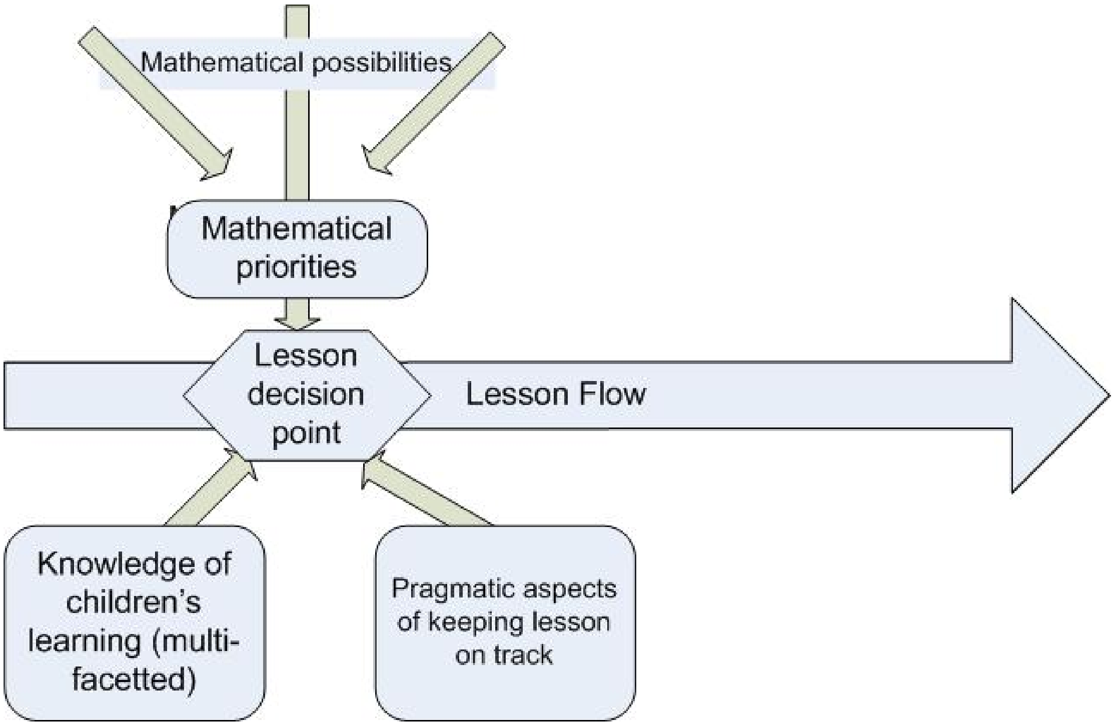

Figure 2. *Decision points in lessons are influenced by many factors.*

Greg's lessons on the Spinners Game

In Greg's class, students played the spinners game at the end of a lesson, and students put forward various ideas about whether the game was fair. During this time, Greg decided that the next lesson should be spent on finding the sample space. Greg then devoted nearly half of his second lesson to an exploration of the sample space. As reported in Chick and Baker (2005) he tightly guided the students in recording all the outcomes and could not deal with alternative approaches. He asked the students to calculate the probabilities of particular outcomes, which was helpful in highlighting the value of enumerating the sample space, but detracted from the problem of ascertaining whether even or odd outcomes were more likely. Students eventually obtained the "40 odds and 41 evens" conclusion, at which point Greg stated that because the "evens" outcome was more likely the game was unfair. There was, however, no discussion of the narrowness of the margin, or the difficulty of confirming this result empirically through the 'first to ten' aspect of the game.

In summary, Greg was much more thorough than Irene in his consideration of sample space, but also very directive. Neither teacher seemed aware of all that the game afforded in advance of using it, as evidenced by the way it was used, although Greg recognised the scope for examining sample space part way through the first lesson. Both teachers were, however, able to bring out some of the concepts in their use of the game, with Irene having a good discussion of the meaning of fairness and the magnitude of the bias, and Greg illustrating sample space and the probability of certain outcomes. An important observation needs to be made here. The teacher guide that was the source of the activity gave too little guidance about what the spinner game afforded and how to bring it out. Even if such guidance had been provided, there is also still the miniscule bias problem inherent in the game's structure that affects what the activity can afford. It is very difficult to convincingly make some of the points about sample space, likelihood, and fairness with the example as it stands. It can be done, but the activity probably needs to be supplemented with other examples that make some of the concepts more obvious (see, e.g., Baker & Chick, 2007). This highlights the crucial question of how can teachers be helped to recognise what an example affords and then adapt it, if necessary, so that it *better* illustrates the concepts that it is intended to convey.

Interestingly, in both classes the students did not—indeed could not in any reasonable time frame —play the game long enough for the slight unfairness to be genuinely evident in practice, yet most students claimed that the game was biased towards even. This may have occurred because the incorrect parity argument made them more aware of the even outcomes than the odd ones.

The observers were surprised by the tight way in which Greg controlled the method by which the outcomes were enumerated. He wanted to see the 81 outcomes, along the lines of the enumeration on the left hand side of Figure 3, although in an array setting out. Greg seemed constrained by his mathematical knowledge, having only one way to think of the sample space—via exhaustive enumeration. When a student offered an erroneous suggestion which could have been readily adapted to a more elegant and insightful method, he did not encourage or discuss it. In fact, there are many bridges between the totally routine method of writing out 81 outcomes and counting how many totals are even or odd, and insightful ways which give the answer quickly. At the top right hand side of Figure 3, for example, is one of the bridges. As they begin work on the exhaustive enumeration on the left hand side, students might be encouraged to note the patterns – alternating evens and odds for a fixed first choice, the EOEOEOEOE pattern when the first choice is odd and the OEOEOEOEO pattern when the first choice is even. These patterns are easily explained by students, and they can be readily utilised to find the how many even and odd sums there are, either by addition or by multiplication as outlined in the figure. The tree diagram approach at the bottom of Figure 3 would be too sophisticated for Greg's young students, since it relies on more strongly combinatorial thinking, but a version of it might be reached after experience with the patterns above.

Enumeration of 81 separate outcomes			
First	**Second**	**Sum**	**Sum**
1	1	2	E
1	2	3	O
1	3	4	E
1	4	5	O
1	5	6	E
1	6	7	O
1	7	8	E
1	8	9	O
1	9	10	E
2	1	3	O
2	2	4	E
2	3	5	O
2	4	6	E
2	5	7	O
2	6	8	E
2	7	9	O
2	8	10	E
2	9	11	O
3	1	4	E
3	2	5	O
3	3	6	E
3	4	7	O
3	5	8	E
etc	etc	…	…

Enumeration mixing number and parity considerations					
First	**Second**	**Sum**	**First**	**Second**	**Sum**
Odd (eg 1)	1	E	Even (eg 2)	1	O
	2	O		2	E
	3	E		3	O
	4	O		4	E
	5	E		5	O
	6	O		6	E
	7	E		7	O
	8	O		8	E
	9	E		9	O
5 evens and 4 odds			4 evens and 5 odds		
The 5 odd numbers contribute 25 evens and 20 odds, and the 4 even numbers contribute 16 evens and 20 odds, making 41 evens and 40 odds altogether.					

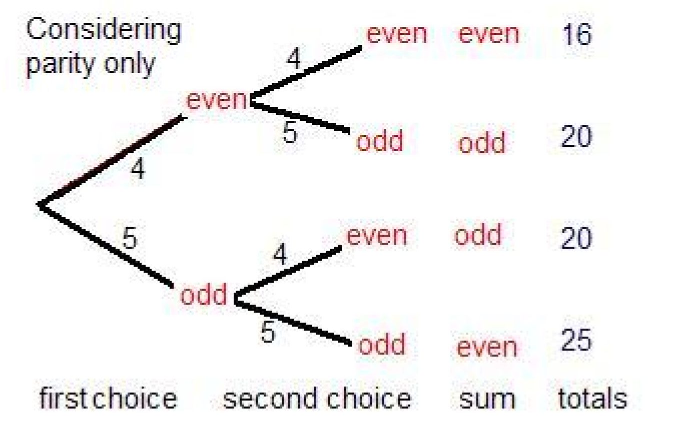

Figure 3. *Three different ways of counting numbers of odd and even spinner totals*

In considering why Greg made his decision to focus his lesson on finding the 81 element sample space in one particular way, it is again likely that his decision is influenced by judgements about mathematical factors, factors related to the students and their current knowledge and pragmatic factors related to the lesson. Greg decided in the first lesson that he would allocate the second lesson to finding the sample space, so it was a priority for him, and he taught it thoroughly. Whereas Irene's treatment of sample space appeared rushed in response to a shortage of time, Greg decided that this was sufficiently important for a second lesson. His priority given to the idea of sample space is also evident in the observation that he did not focus only on the spinner game, but used the sample space to find the probability of events unrelated to the initial spinner game.

As is illustrated in Figure 2, mathematical priorities can only be chosen from the mathematical possibilities that are perceived by the teacher. Consequently, it may be that Greg's focus on one way of finding the sample space was because he was not aware of other ways, or was uncertain of their validity. On the other hand, it may have been a more active prioritising. He may have seen value in teaching students about systematic listing, and wanted students to go through that process very thoroughly, getting a real 'feel' for how to go through the cases one by one. From yet another point of view, Greg may have judged that the full, very routine, case-by-case enumeration was at an appropriate level for his target group in that class, and so he may have selected the method as optimal for the whole class, even if not for each individuals.

This is all speculation, even though Greg was interviewed about his lesson (which contained many other features). It is simply not possible for teachers to thoroughly explain each of the myriad decisions that are made in the course of any one lesson. The point of this discussion, though, is that at any stage in the lesson, Greg was aware of certain mathematical possibilities. These may have resulted from deep or superficial insight into the spinners game; they may be numerous or sparse; they may be mainly procedural or extend to strategic thinking etc. To make a decision on how to respond to a student's question or a mathematical problem arising in the conduct of the class, Greg has to set priorities and act on them. In this way, we see that a teacher's mathematical knowledge (conceptual, procedural, strategic etc) sets the choices and so is very important, but good decision making also depends on teachers being able to make good choices amongst them, in the light of progressing the main aims of the lesson.

Helen's lesson on the spinners game

Even when lessons are videotaped and teachers are interviewed after the lesson, much of the mathematical thinking upon which teachers make decisions about the paths of lessons remains hidden. For this reason, the next example is about a discussion on an episode in a teacher education class, which we discussed together on several occasions.

Helen teaches pre-service primary teacher education students and is a highly accomplished mathematical thinker. She had observed the lessons of Irene and Gary, and decided to use the spinners game in class. She wanted student teachers to analyse what mathematical learning it could generate and how. To simplify and also to extend the game, Helen changed the numbers on the two spinners (see Baker and Chick (2007) for examples).

On one occasion, Helen's class used two spinners labelled with 0, 1, 2 and 3. This small change, selected by Helen to simplify the game, caused a new complication. Many of her students began the enumeration, but halted when they needed to decide whether the sum of 0, obtained by throwing 0 on each spinner, was an even number or an odd number, both or neither. This turned what was intended to be a short mathematical episode using a simplified spinners game used, to an unpredicted query about odd and even numbers.

At this point, Helen faces a decision. Once again it will be informed by her knowledge of mathematics and her mathematical thinking during the lesson, and by weighing the priorities for the lesson. This will be discussed below. However, it is worth observing first that Helen had not predicted the evenness of 0 would be such an obstacle to the progress of this lesson. In future use of the spinners game, having this additional knowledge of students (urther pedagogical content knowledge), she may avoid using the number 0 on the spinners so that the lesson proceeds without this obstacle, or she may deliberately choose it to uncover these misconceptions.

Addressing the apparently simple question of whether 0 is even or odd or neither or both, draws again on mathematical knowledge and pedagogical content knowledge (Shulman, 1986, 1987) working in tandem. The student teachers were very familiar with the fact that 2, 4, 6, 8, 10, 12, etc are even numbers. Why would they query whether 0 is even, and what would convince them that it is? Possibly the reason for the difficulty is that students draw on intuitive meanings for ‘even’, rather than a mathematical definition. For example, they may associate an even number with the possibility of pairing up. If there is an even number of children in our class, we can go for a walk arranged in pairs. If there is an odd number of children, there will be one left over, as illustrated in Figure 4.

Figure 4. *An even number of children can walk in pairs, but not an odd number*.

This informal interpretation of ‘even number’ is difficult to apply to decide whether 0 is even or odd, because whilst there is certainly not ‘one left over’, there are no pairs either. Kaplan (1999) discusses difficulties like this. Alternatively, students who draw just on the list of examples to decide if a number is even or odd (2, 4, 6, … etc) have no way of knowing whether 0 should really be on the list or not, when there is no principle to guide them. They know 0 is special – is this another way in which it is special?

Helen was keen to draw her students’ attention to the mathematical definition of an even number, but she reported that she immediately saw two possibilities. She could say that an even number is defined to be an integer which is exactly divisible by 2 or that an even number is defined to be an integer that is equal to 2 times an integer. This might seem a small difference, but Helen chose the second version because of her previous experience with the awkwardness in teaching associated with discussions of dividing *by* zero. Even though the test for evenness does not involve dividing *by* zero (but dividing *by* 2), Helen avoided the division explanation because she felt students may confuse the situations. In other words, she presented students with finding whether there is an integer satisfying the first rather than the (equivalent) second equation below:

$$0 = 2 \times ?$$

$$0 \div 2 = ?$$

[In neither case could she avoid the likely obstacle of students' uncertainly about whether 0 is an integer.] Here we see that Helen's strong mathematical knowledge and her ability to see the mathematical possibilities quickly presented her with possibilities. Her pedagogical content knowledge (in this case of likely students' difficulties) guided her choice.

Was it best to pause to discuss why 0 is even? Helen could have just asserted that 0 is even and moved the lesson back on the track of investigating the fairness of the spinners game. When reflecting on this question, Helen asserted that the diversion was useful because it enabled her to clarify some fundamental misunderstandings about zero and to show how mathematical concepts are determined by definitions. Here, we see that Helen justifies her choice in terms of her understanding of important principles of doing mathematics – in this case the role of definitions in mathematics. More fundamentally, it seems to reveal a predisposition on Helen's part to avoid having students see mathematics as arbitrary and without reason.

After her observations of the lessons of Irene and Greg, Helen and her colleague published a suggested teaching sequence for primary classes using the spinners game (Baker and Chick, 2007). The spinners she suggests have no zeros. Her suggested sequence begins with a pair of spinners each with just 3 digits, arranged so that there is a strong enough bias to be evident in empirical trials. Students begin by finding this empirical experience of the bias, tallying class results. Students then draw up the sample space and compare theoretical probabilities to empirical class results. They discuss variations between theory and experiment. The pair of spinners chosen are biased towards odd totals (they do not have the same numbers on each spinner – see mathematical note below. Helen has selected these spinners so that the false parity arguments give an obviously wrong answer. This is a very substantial example of mathematical thinking being used in lesson planning, again in concert with pedagogical content knowledge – in this case knowledge of students' false arguments. Helen's suggested lesson sequence then moves back to the original spinners problem. this gives experience in finding a large sample space systematically and subtleties of comparing theoretical and empirical results when the bias is small. Finally students create their own spinners and discuss what they designed the spinners for, how unfair the game is, and what is likely to happen if they play the game many times.

Conclusion

At the beginning of this paper, I drew an analogy between teaching a mathematics lessons and solving a real world problem with mathematics. I noted that in order to use mathematics to solve a problem in an area of application, mathematics must be used in combination with knowledge from the area of application. In the case of teaching mathematics, the area of application is the classroom and so the teacher as 'mathematical problem solver' has to draw on general pedagogy as well as mathematical pedagogical content knowledge to contribute to the solution. As will many problems in areas of application of mathematics, these teaching problems need to be solved in an environment that is rich in constraints: short lesson times, inadequate resources at hand, etc. In the teachers' role of analysing subject matter, designing curricula or in creating a plan for a good lesson, solving the problem can occur with adequate time for reflection, testing ideas and reconsidering choices. However, in the course of a lesson, this mathematico-pedagogical thinking happens on a minute-by-minute basis, with the aim of responding to students in a mathematically productive way. If teachers are to encourage mathematical thinking in students, then they need to engage in mathematical thinking throughout the lesson themselves, but this mathematical thinking is under severe time pressure.

In the conduct of a lesson, teachers see various mathematical possibilities. Some teachers will see more than others in any given situation and some of the possibilities that teachers see may not be correct. The process of choosing amongst these possibilities, which again occurs on a minute by minute basis, will be guided by the deep knowledge of the students (the actual current mathematical knowledge of these students as well as thinking typical of students like these), operating under the constraints of teaching a lesson in a fixed time to achieve an identified goal. Teachers who are stronger mathematical thinkers will see more possibilities, and in the moment when a decision needs to be made, their choices will be better informed by teaching underlying mathematical processes and strategies.

A mathematical note

Solving the problem of bias in the spinners game is a nice example in algebraic factorisation, with surprising results.

If there are n even numbers and m odd numbers on the spinners, then there are $n^2 + m^2$ ways of getting an even total, and $2mn$ ways of getting an odd total (see Figure 5). Since $n^2 + m^2 - 2mn = (n-m)^2 \geq 0$ we can conclude that

(i) if $n = m$ then the spinner game is fair

(ii) otherwise, there is always slightly more chance of getting an even number.

Moreover, if the numbers on the spinners are consecutive whole numbers, then n and m will either be equal or differ by 1 (ie n –m = 0 or |n-m| = 1). This means that the number of even sums will always be equal to, or one more than the number of odd sums. In this way, we see that the very close comparative probabilities of the original spinners game (41/81 and 40/81) are typical of having consecutive numbers on the spinners.

To generalise further, if there are n_1 evens and m_1 odds on the first spinner and n_2 and m_2 on the second spinner (respectively) then there are $n_1n_2 + m_1m_2$ even sums and $n_1m_2 + n_2m_1$ odd sums. Are evens or odds more likely to be thrown? Calculate the difference in number of outcomes:

$$n_1m_1 + n_2m_2 - (n_1m_2 + n_2m_1) = (n_1 - m_1)(n_2 - m_2)$$

This means that if evens are more prevalent on both spinners OR odds are more prevalent on both spinners (ie the two factors in the final product have the same sign), then the game is biased in favour of the even sums. Alternatively, it evens are more prevalent on one spinner and odds more prevalent on the other spinner (ie the two factors in the product have opposite signs), then the game is biased in favour of odds.

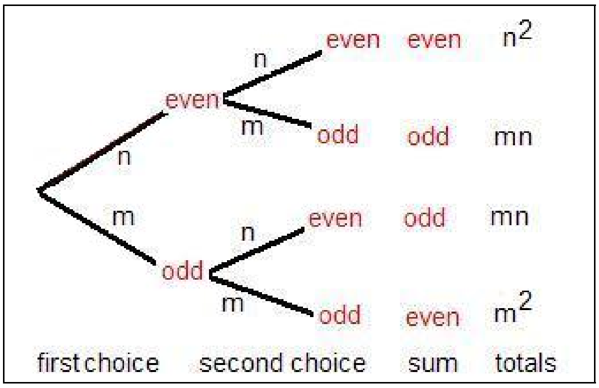

Figure 5. *Odd and even spinner totals from spinners with n even and m odd numbers*

References

APEC –Tsukuba (Organising Committee) (2006) *First announcement. International Conference on Innovative Teaching of Mathematics through Lesson Study.* CRICED, University of Tsukuba.

Baker, M., & Chick, H. L. (2006). Pedagogical content knowledge for teaching primary mathematics: A case study of two teachers. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 60-67). Sydney: MERGA.

Baker, M., & Chick, H. L. (2007). Making the most of chance. *Australian Primary Mathematics Classroom*, 12(1), 8-13.

Chick, H. L., & Baker, M. (2005). Teaching elementary probability: Not leaving it to chance. In P. C. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia*, pp. 233-240). Sydney: MERGA..

Chick, H.L. (2007) Teaching and Learning by example. In J. Watson & K. Beswick (Eds), *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia*, Hobart, Tasmania: MERGA.

Feely, J. (2003). *Nelson maths for Victoria: Teacher's resource Year 5*. Melbourne: Thomson Nelson.

Kaplan, R. (1999) *The Nothing That Is: A Natural History of Zero.* London: Penguin

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Shulman, L.S. (1986) Those who understand: Knowledge growth in teaching, *Educational Researcher* 15 (2), 4-14.

Shulman, L.S. (1987) Knowledge and teaching: Foundations of the new reform, *Harvard Educational Review* 57(1), 1-22.

Stacey, K. (2007) What is mathematical thinking and why is it important? *APEC Symposium. Innovative teaching mathematics through lesson study II*. 3-4 December 2006.

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